98學年度第1學期 微積分甲一組期中考解答

1. (18%) (a) Let $f_1(x) = x^{a^a} + a^{x^a} + a^{a^x},\ a > 0,\ x > 0$, and $f_2(x) = \int_{x^x}^{\sqrt{\ln x}} e^{t^2}\, dt,\ x > 1$.

Then $f_1'(x) =$ ____ ,

$f_2'(x) =$ ____ .

(b) Suppose that $y = y(x)$ is implicitly defined by $x^y = y^x$.

Then at $(x, y) = (2, 4)$, $\dfrac{dy}{dx} =$ ______________________________ .

Sol:

(a-1) First note that :

$$dx^b/dx = bx^{b-1}$$

&

$$db^x/dx = (\ln b)b^x$$

We use this and chain rule to solve the problem.

$$\begin{aligned} df_1(x)/dx &= d(x^{a^a})/dx + d(a^{x^a})/dx + d(a^{a^x})/dx \\ &= a^a x^{a^a-1} + (\ln a)a^{x^a} d(x^a)/dx + (\ln a)a^{a^x} d(a^x)/dx \\ &= a^a x^{a^a-1} + (\ln a)a^{x^a} a(x^{a-1}) + (\ln a)a^{a^x}(\ln a)a^x \\ &= a^a x^{a^a-1} + (\ln a)a^{x^a+1}(x^{a-1}) + (\ln a)^2 a^{a^x+x} \end{aligned}$$

(a-2)

$$\begin{aligned} df_2(x)/dx &= d\int_{x^x}^{\sqrt{\ln x}} f(t)dt/dx \\ &= d\int_{x^x}^{a} e^{t^2} dt/dx + d\int_{a}^{\sqrt{\ln x}} e^{t^2} dt/dx \\ &= (d\int_{x^x}^{a} e^{t^2} dt/dx^x)dx^x/dx + (d\int_{a}^{\sqrt{\ln x}} e^{t^2} dt/d\sqrt{\ln x})d\sqrt{\ln x}/dx \\ &= (-e^{x^{x2}})(x^x)(\ln x + 1) + (e^{\sqrt{\ln x}^2})(1/2)((\ln x)^{-1/2})(1/x) \\ &= -e^{x^{2x}} x^x(\ln x + 1) + \frac{1}{2}(\ln x)^{-1/2} \end{aligned}$$

$$\left(\frac{dx^x}{x} = \frac{de^{x\ln x}}{dx} = e^{x\ln x}\frac{dx\ln x}{dx} = x^x(\ln x + 1)\right)$$

(b) Implicit function:

$$x^y = y^x$$

First take "ln" to both sides

$$y \ln x = x \ln y$$

Differentiate both sides w.r.t x (take y as a function of x)

$$\begin{aligned}\frac{dy}{dx}\ln x + \frac{y}{x} &= \ln y + x\frac{dy/dx}{y} \\ \Rightarrow \frac{dy}{dx} &= \frac{\ln y - y/x}{lnx - x/y}\end{aligned}$$

So

$$\begin{aligned}\frac{dy}{dx}|_{(2,4)} &= \frac{\ln 4 - 2}{\ln 2 - 1/2} \\ &= \frac{2\ln 4 - 4}{2\ln 2 - 1} \quad or \quad \frac{4(\ln 2 - 1)}{\ln 4 - 1} \quad or \quad \frac{\ln 16 - 4}{\ln 4 - 1}\end{aligned}$$

2. (18%) Determine whether the following limits exist. If the limit exists, evaluate it. If the limit doesn't exist, explain why.

(a) $\lim_{x\to 0}(1 + |\sin x|)^{\frac{1}{x}} =$ ______________________________ ,

(b) $\lim_{x\to 0}\left(\frac{2^x + 3^x + 5^x}{3}\right)^{\frac{1}{x}} =$ ______________________________ .

(c) First express $\sum_{k=1}^{n}\frac{\ln 2 + \ln(n+k) - \ln n}{n+k}$ as a Riemann sum for a function defined on $[0, 2]$.

Then evaluate $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\ln 2 + \ln(n+k) - \ln n}{n+k}$.

Answer. $\sum_{k=1}^{n}\frac{\ln 2 + \ln(n+k) - \ln n}{n+k} =$ ______________________________ ,

$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\ln 2 + \ln(n+k) - \ln n}{n+k} =$ ______________________________ .

Sol:

(a) using the continuity of exponential function and L.H rule

$$\lim_{x\to 0^+}(1 + |\sin(x)|)^{\frac{1}{x}} = \lim_{x\to 0^+}\exp(\frac{1}{x}\ln(1 + |\sin(x)|))$$

$$= \exp(\lim_{x\to 0^+}(\frac{\cos(x)}{1+|\sin(x)|})) = \exp(1)$$

$$\lim_{x\to 0^-}(1+|\sin(x)|)^{\frac{1}{x}} = \lim_{x\to 0^-}\exp(\frac{1}{x}\ln(1+|\sin(x)|))$$

$$= \exp(\lim_{x\to 0^-}(\frac{-\cos(x)}{1-\sin(x)})) = \exp(-1)$$

so the limit doesn't exist .

(b) still using the continuity of exponential function and L.H rule

$$\lim_{x\to 0^+}\left(\frac{2^x+3^x+5^x}{3}\right)^{\frac{1}{x}} = \lim_{x\to 0^+}\exp(\frac{1}{x}\ln(\frac{2^x+3^x+5^x}{3}))$$

$$= \exp(\lim_{x\to 0^+}(\frac{\frac{2^x\ln(2)+3^x\ln(x)+5^x\ln(x)}{3}}{\frac{2^x+3^x+5^x}{3}})) = \exp(\frac{\ln(2)+\ln(3)+\ln(5)}{3})$$

$$= \exp(\frac{\ln(30)}{3}) = 30^{\frac{1}{3}}$$

the same argument for

$$\lim_{x\to 0^-}\left(\frac{2^x+3^x+5^x}{3}\right)^{\frac{1}{x}}$$

so the limit exists and the limit is $30^{\frac{1}{3}}$

(c)

$$\sum_{k=1}^{n}\frac{\ln 2+\ln(n+k)-\ln(n)}{n+k} = \sum_{k=1}^{n}\frac{\ln(2+\frac{2k}{n})}{n+k}$$

$$= \sum_{k=1}^{n}\frac{2}{n}\frac{\ln(2+\frac{2n}{k})}{2+\frac{2k}{n}} or \sum_{k=1}^{n}\frac{1}{n}\frac{\ln 2(1+\frac{k}{n})}{1+\frac{k}{n}}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\ln 2+\ln(n+k)-\ln(n)}{n+k} = \int_0^2\frac{\ln(2+x)}{2+x}dx \quad or \quad \int_0^1\frac{\ln(2(1+x))}{1+x}dx$$

$$= \int_0^2 \ln(2+x)d(\ln(2+x)) = \frac{3}{2}(\ln(2))^2$$

3. (8%) Let

$$H(x) = \begin{cases} e^{\frac{1}{x}} & , x<0 \\ m & , x=0 \\ a\sin x + b\cos x + cx & , x>0 \end{cases}$$

Find conditions of m, a, b, c, such that, respectively,

(a) $H(x)$ is continuous everywhere. Answer: ______________________________,

(b) $H(x)$ is differentiable everywhere. Answer: ______________________________,

(c) $H(x)$ has an inflection point at $x = 0$. Answer: ______________________________.

Sol:

(a) Since $H(x)$ is continuous everywhere, thus we have

$$\lim_{x\to 0^+} H(x) = \lim_{x\to 0^-} H(x) = H(0)$$

$\Longrightarrow \lim_{x\to 0^-} e^{\frac{1}{x}} = H(0) = \lim_{x\to 0^+} (a\sin x + b\cos x + cx)$

and $\lim_{x\to 0^-} e^{\frac{1}{x}} = 0$, $H(0) = m$,

$\lim_{x\to 0^+} (a\sin x + b\cos x + cx) = b$

so $b = m = 0$.

(b) Since H(x) is differentiable everywhere, thus $H(x)$ is continuous everythere, so $b = m = 0$

and $\lim_{h\to 0^-} \frac{H(h)-H(0)}{h-0} = H^{'}(0) = \lim_{h\to 0^+} \frac{H(h)-H(0)}{h-0}$

so

$$\begin{aligned}\lim_{h\to 0^-} \frac{e^{\frac{1}{h}}}{h} &= \lim_{h\to 0^-} \frac{\frac{1}{h}}{e^{-\frac{1}{h}}} \\ &= \lim_{h\to 0^-} \frac{-\frac{1}{h^2}}{\frac{1}{h^2}e^{-\frac{1}{h}}} \\ &= -\lim_{h\to 0^-} e^{\frac{1}{h}} = 0\end{aligned}$$

and $\lim_{h\to 0^+} \frac{a\sin h + ch}{h} = a + c$

thus the conditions are $b = m = 0$, $a + c = 0$.

(c) Since $H(x)$ is continuous at $x = 0$, thus $b = m = 0$

For $x \lneq 0$,

$H^{'}(x) = -\frac{1}{x^2}e^{\frac{1}{x}}$ and $H^{"}(x) = \frac{(2x+1)e^{\frac{1}{x}}}{x^4}$,

so $H^{"}(x) \gneq 0$ for $-\frac{1}{2} < x < 0$.

For $x \gneq 0$, $H^{'}(x) = a\cos x + c$ and $H^{"}(x) = -a\sin x$.

Since $x = 0$ is an inflection point and $H^{"}(x) \gneq 0$ for $x \in (-\dfrac{1}{2}, 0)$, hence $a \gneq 0$

thus the conditions are $b = m = 0$, $a \gneq 0$

4. (12%) Let $g(x) = \dfrac{a}{x^3+3x+4} + \dfrac{b}{x^3+x-2}$, $ab > 0$. Show that $g(x) = 0$ has exactly one real solution.

Sol:

$g(x) = \frac{a(x^3+x-2)+b(x^3+3x+4)}{(x^3+3x+4)(x^3+x-2)}$

Define $f(x) = a(x^3+x-2) + b(x^3+3x+4)$.

$x^3+x-2 = (x-1)(x^2+x+2)$; $x^3+3x+4 = (x+1)(x^2-x+4)$

$(x^2+x+2 > 0,\ x^2-x+4 > 0,\ \forall x \in \mathbb{R})$

$f(1) = 8b \neq 0$; $f(-1) = -4a \neq 0$

So $f(x) = 0$ and $g(x) = 0$ have the same roots.

Because $f(1)f(-1) = -32ab < 0$, there is at least one root of $f(x) = 0$ in $(-1, 1)$ by IVT.

If $f(\alpha) = f(\beta) = 0$ for some $\alpha < \beta$.

By MVT, $\exists\ c \in (\alpha, \beta)$ such that

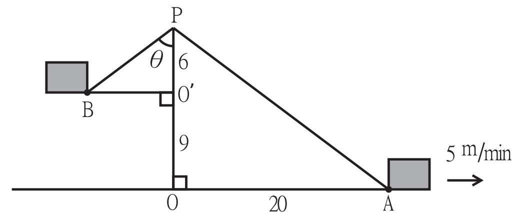
$$f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha} = 0.$$

But $f'(c) = a(3c^2+1) + b(3c^2+3) > 0$ (or < 0, depending on the signs of a and b), a contradiction.

So $f(x) = 0$ has exactly one root. $\Longrightarrow g(x) = 0$ has exactly one root.

5. (12%) Car A at the lower level pulls car B, which is located on the upper level 9 meters higher, with constant velocity 5 m/min to the right while a pulley 6 meters above the upper level is used to connect the two cars. Suppose that the total length of the rope is 35 meters. Let O be the point right underneath pulley P on the lower level, and $\theta = \angle BPO$. Find the changing rate of θ when car A is 20 meters away from point O. See figure below.

Answer: ________________________________ .



Sol:

Let $\overline{OA} = x(t)$, then we have

$$35 - \sqrt{225 + x(t)^2} = 6 \sec \theta(t)$$

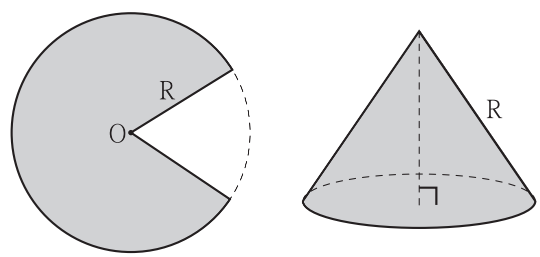
Differentiate both sides with respect to t and we get

$$-\frac{2x(t) \cdot x'(t)}{2\sqrt{225 + x(t)^2}} = 6 \sec \theta(t) \tan \theta(t) \cdot \theta'(t)$$

Now since $x'(t) = 5$ and when $x(t^0) = 20$, we have $\sec \theta(t^0) = \frac{5}{3}$ and $\tan \theta(t^0) = \frac{4}{3}$, thus we get $\theta'(t^0) = -\frac{3}{10}$ (rad/min).

6. (12%) A sector is cut off from a circle of radius R, $R > 0$. The remaining part (shaded region) is used to construct a right circular cone. Find the maximal possible volume of the cone. See figure below. (Hint. The volume of a right circular cone $= \frac{1}{3}$(base area)·(height).)

Answer: The maximal volume= ____________________________________.



O : center of the circle

Sol:

Assume that the height of the cone is equal to $h > 0$. Then the radius of the circle at the bottom is equal to $\sqrt{R^2 - h^2}$. The volume of the cone is

$$V(h) = \frac{\pi}{3}\left(\sqrt{R^2 - h^2}\right)^2 h = \frac{\pi}{3}\left(R^2 h - h^3\right)$$

To find the maximum value of $V(h)$, consider $V'(h) = 0$. This gives

$$\frac{\pi}{3}\left(R^2 - 3h^2\right) = 0$$

So $h^2 = R^2/3$. That is, $h = R/\sqrt{3}$ since $R > h > 0$. Now we can check this is exactly the point which makes V occur a maximum value by considering $V(0) = V(R) = 0$ and $V(h) \geq 0$

trivially. And the maximum value is

$$V\left(\frac{R}{\sqrt{3}}\right) = \frac{2\sqrt{3}\pi R^3}{27}$$

7. (20%) Let $f(x) = \dfrac{x^2(x-2)}{(x+1)^2}$. Answer the following questions.

(a) The domain of $y = f(x)$ is ____________________ .

(b) $f'(x) =$ ____________________ .

(c) $y = f(x)$ has critical point(s) at ____________________ .

(d) $f''(x) =$ ____________________ .

(e) $y = f(x)$ is increasing on interval(s) ____________________ ,

$y = f(x)$ is decreasing on interval(s) ____________________ .

(f) $y = f(x)$ is concave up on interval(s) ____________________ ,

$y = f(x)$ is concave down on interval(s) ____________________ .

(g) Find the (x, y)-coordinates of the following points if exist.

Local maximum point(s) : ____________________ ,

Local minimum point(s) : ____________________ ,

Inflection point(s) : ____________________ .

(h) Find the asymptotes of the graph of $y = f(x)$ if exist.

Vertical asymptotes(s) : ____________________ ,

Horizontal asymptotes(s) : __________ as $x \to$ __________ ,

Slant asymtotes(s) : __________ as $x \to$ __________ .

(i) Sketch the graph of $y = f(x)$ on page 11.

Sol:

Let $f(x) = \dfrac{x^2(x-2)}{(x+1)^2}$. Answer the following questions.

(a) The domain of $y = f(x)$ is $\underline{\mathbb{R} \setminus \{-1\}}$.

(b) $f'(x) = \underline{\dfrac{x(x+4)(x-1)}{(x+1)^3}}$.

(c) $y = f(x)$ has criticial point(s) at $\underline{x = -4, 0, 1}$.

(d) $f''(x) = \underline{\dfrac{2(7x-2)}{(x+1)^4}}$.

(e) $y = f(x)$ is increasing on interval(s) $\underline{(-\infty, 4) \cup (-1, 0) \cup (1, \infty).}$

$y = f(x)$ is decreasing on intervals(s) $\underline{(-4, -1) \cup (0, 1).}$

(f) $y = f(x)$ is concave up on interval(s)$\underline{(\frac{2}{7}, \infty)}$.

$y = f(x)$ is concave down on interval(s) $\underline{(-\infty, -1) \cup (-1, \frac{2}{7})}$.

(g) Find the $(x, y)-$coordinates of the follwing points if exist.

Local maximal point(s): $\underline{(-4, \frac{32}{3}), (0, 0).}$

Local minimal point(s): $\underline{(1, -\frac{1}{4})}$.

Inlfection point(s): $\underline{(\frac{2}{7}, -\frac{16}{189})}$.

(h) Find the asymptotes of the graph of $y = f(x)$ if exist.

Vertical asymptotes(s): $\underline{x = -1}$.

Horizontal asymptotes(s): $\underline{\textit{none}}$.

Slant asymptotes(s):$\underline{y = x - 4}$ as $x \to \underline{\pm\infty}$.

(i)