98學年度第1學期 微積分甲二組期末考解答

1. (10%) Evaluate the integral $\int \frac{dx}{x^3-1}$.

   Sol:

Step 1. Apply the partial fraction decomposition of the integrand, we have

$$\frac{1}{x^3-1} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1}.$$

Determine the coefficients a, b, c by the following identity

$$1 = a(x^2+x+1) + (bx+c)(x-1).$$

Note that $a+b=0$. Let $x=1$, we have $1=3a$. Hence,

$$a = \frac{1}{3} \quad \text{and} \quad b = -\frac{1}{3}.$$

Set $x=0$, we have $1 = a-c$ or $c = -\frac{2}{3}$.

Step 2. Evaluate the integral $a\int \frac{1}{x-1}dx$.

Note that

$$\frac{1}{3}\int \frac{1}{x-1}dx = \frac{1}{3}\ln|x-1| + C_1.$$

Step 3. Evaluate the integral $\int \frac{bx+c}{x^2+x+1}dx$ by writing

$$\frac{1}{3}\int \frac{x+2}{x^2+x+1}dx = \frac{1}{6}\int \frac{2x+1}{x^2+x+1}dx + \frac{1}{2}\int \frac{1}{x^2+x+1}dx.$$

Note that

$$\frac{1}{6}\int \frac{2x+1}{x^2+x+1}dx = \frac{1}{6}\ln(x^2+x+1) + C_2.$$

Write

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

and denote $x+\frac{1}{2}$ by $\frac{\sqrt{3}}{2}\tan\theta$. Hence,

$$x^2+x+1 = \frac{3}{4}(\tan^2\theta+1) = \frac{3}{4}\sec^2\theta$$

and $dx = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$.

$$\frac{1}{2}\int \frac{1}{x^2+x+1}dx = \frac{1}{2}\frac{2}{\sqrt{3}}\int d\theta = \frac{1}{\sqrt{3}}\theta + C_3 = \frac{1}{\sqrt{3}}\tan^{-1}\theta + C_3 = \frac{1}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C_3.$$

$$\int \frac{dx}{x^3-1} = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C.$$

2. (10%) Evaluate the integral $\int \frac{x}{\sqrt{x^2+2x+2}}\,dx$.

Sol:

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+2x+2}}dx &= \int \frac{x+1}{\sqrt{(x+1)^2+1}} - \frac{1}{\sqrt{(x+1)^2+1}}dx \\ &= \int \frac{u}{u^2+1}du - \int \frac{du}{\sqrt{u^2+1}} \\ &= \sqrt{u^2+1} - ln\left|u+\sqrt{1+u^2}\right| + c \\ &= \sqrt{x^2+2x+2} - ln\left|(x+1)+\sqrt{x^2+2x+2}\right| + c\end{aligned}$$

Where the 3rd equality follows by:

$$u = \tan\theta;\ du = \sec\theta;\ \ \tan^2\theta + 1 = sec^2\theta$$

$$\begin{aligned}\int \frac{\sec^2\theta}{\sec\theta}d\theta &= \int \sec\theta d\theta \\ &= ln\left|\sec\theta + \tan\theta\right| + c \\ &= ln\left|u+\sqrt{1+u^2}\right| + c\end{aligned}$$

3. (10%) Let $I_n = \int_0^\infty x^n e^{-x}\,dx$.

   (a) Find the recursive relation between I_n and I_{n-1}.

   (b) Compute I_3.

   (c) Find the general formula of I_n.

Sol:

(a)

$$\begin{aligned} I_n &= \int_0^\infty x^n e^{-x} dx = \lim_{a\to\infty} \int_0^a x^n e^{-x} dx \\ &= \lim_{a\to\infty} -x^n e^{-x}\Big|_0^a + n\int_0^a x^{n-1} e^{-x} dx \\ &= \lim_{a\to\infty} -x^n e^{-x}\Big|_0^a + nI_{n-1} \end{aligned}$$

Since $\lim_{a\to\infty} -x^n e^{-x}\Big|_0^a = \lim_{a\to\infty} \frac{a^n}{e^a} = 0$

Hence $I_n = nI_{n-1}$

(b) Follow (a) we have $I_3 = 3I_2 = 3\cdot 2I_1 = 3\cdot 2\cdot 1 I_0$

Moreover, we have $I_0 = \int_0^\infty e^{-x} dx = \lim_{a\to\infty} \int_0^a e^{-x} dx = \lim_{a\to\infty} -e^{-x}\Big|_0^a = 1$

Hence $I_3 = 3! = 6$.

(c) Same discuss as (b) we can get the general formula of I_n

$I_n = nI_{n-1} = n\cdot(n-1)\cdot I_{n-1} = \cdots = n! I_0 = n!$

4. (15%) Given $0 < a < b$, find the arc length of $y = \ln\frac{e^x+1}{e^x-1}$ for $a \le x \le b$.

Sol:

$$\begin{aligned} \frac{dy}{dx} &= \frac{(\frac{e^x+1}{e^x-1})'}{\frac{e^x+1}{e^x-1}} = \frac{\frac{e^x(e^x-1)-e^x(e^x+1)}{(e^x-1)^2}}{\frac{e^x+1}{e^x-1}} \\ &= \frac{-2e^x}{(e^x-1)^2} \times \frac{e^x-1}{e^x+1} = \frac{-2e^x}{e^{2x}-1} \end{aligned}$$

$$\begin{aligned} L &= \int_a^b \sqrt{1+(\frac{dy}{dx})^2} dx = \int_a^b \sqrt{1+(\frac{-2e^x}{e^{2x}-1})^2} dx \\ &= \int_a^b \sqrt{\frac{e^{4x}+2e^{2x}+1}{(e^{2x}-1)^2}} dx = \int_a^b \frac{e^{2x}+1}{e^{2x}-1} dx \end{aligned}$$

Let $t = e^x$, then $\frac{dt}{dx} = e^x \to dx = \frac{dt}{t}$.

$x = a \to t = e^a;\ x = b \to t = e^b$

$$\begin{aligned} L &= \int_{e^a}^{e^b} \frac{t^2+1}{t(t+1)(t-1)} dt \\ &= \int_{e^a}^{e^b} (\frac{-1}{t} + \frac{1}{t+1} + \frac{1}{t-1}) dt \\ &= (-\ln t + \ln(t+1) + \ln(t-1))\Big|_{e^a}^{e^b} \\ &= \ln \frac{e^{2b}-1}{e^{2a}-1} + a - b \end{aligned}$$

5. (10%) The region in the first quadrant enclosed by $x = y^2$, x-axis and $x = 1$ is rotated about the line $y = 3$. Find the volume of the solid.

Sol:

(1) Disk Method:

$$\begin{aligned} \text{Area} &= \int_0^1 \pi \cdot 3^2 - \pi(3-\sqrt{x})^2 dx \\ &= \int_0^1 6\pi\sqrt{x} - \pi x dx \\ &= 4\pi x^{3/2} - \frac{\pi}{2}x^2 \Big|_0^1 \\ &= \frac{7}{2}\pi \end{aligned}$$

(2) Shell Method:

$$\begin{aligned} \text{Area} &= \int_0^1 2\pi(3-y)(1-y^2) dy \\ &= \int_0^1 2\pi(3 - y - 3y^2 + y^3) dy \\ &= 2\pi \left(3y - \frac{y^2}{2} - y^3 + \frac{y^4}{4}\right) \Big|_0^1 \\ &= \frac{7}{2}\pi \end{aligned}$$

6. (10%) Find the orthogonal trajectories of $\{y^2 = kx^3,\ k \in \mathbb{R}\}$ which passes through the point $(1, 2)$.

Sol:

$\dfrac{dy}{dx} = \dfrac{-2x}{3y}$

$-x^2 + c = \frac{3}{2}y^2$
$\frac{3}{2}y^2 = -x^2 + 7$

7. (10%) (a) Find the particular solution of $3xy' - y = \ln x + 1,\ x > 0$, satisfying $y(1) = -2$.

(b) Find the value $y(e)$.

Sol:

(a) $y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}$
integrating factor $V = e^{\int -\frac{1}{3x}dx} = x^{-\frac{1}{3}}$

then

$$\begin{aligned} x^{-\frac{1}{3}}y &= \frac{1}{3}\int (\ln x + 1)x^{-\frac{4}{3}}dx \\ &= -x^{-\frac{1}{3}}(\ln x + 1) + \int x^{-\frac{4}{3}}dx \quad \text{(integration by part)} \\ &= -x^{-\frac{1}{3}}(\ln x + 1) - 3x^{-\frac{1}{3}} + C \end{aligned}$$

So $y = -(\ln x + 4) + Cx^{\frac{1}{3}}$.

When $x = 1, y = 2$ then $-2 = -4 + C$ so $C = 2$

i.e. $y = 2x^{\frac{1}{3}} - \ln x - 4$

(b) $y(e) = 2e^{\frac{1}{3}} - 5$

8. (10%) Given $a > 0$, one arch of the cycloid $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),\ 0 \le \theta \le 2\pi$, is rotated about the x-axis. Find the area of the resulting surface.

Sol:

$$
\begin{aligned}
A &= \int_0^{2\pi} 2\pi a(1-\cos\theta)\sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta} \\
&= \int_0^{2\pi} 2\pi a(1-\cos\theta)\sqrt{2}a\sqrt{1-\cos\theta} \\
&= 2\pi a^2 \int_0^{2\pi} 4\sin^3(\frac{\theta}{2})d\theta \\
&= 16\pi a^2 \int_0^{\pi} \sin^3\alpha d\alpha \\
&= 16\pi a^2 \int_0^{\pi} (1-\cos^2\alpha)\sin\alpha d\alpha \\
&= 16\pi a^2 \int_{-1}^{1} (1-u^2)du \\
&= 16\pi a^2 \frac{4}{3} \\
&= \frac{64}{3}\pi a^2
\end{aligned}
$$

9. (15%) (a) Plot the region A which is inside the circle $r = 6\cos\theta$ and outside the cardioid $r = 2(1+\cos\theta)$.

(b) Find the area of A.

(c) Find the length of the boundary of the region A.

Sol:

(a) The graph $r = 6\cos\theta$ is a circle with radius 3 and center $(3, 0)$

The graph $r = 2 + 2\cos\theta$ is shaped like a heart symmetric at x-axis

The graph outside the cardiod and inside the circle is shaped like moon

(b) First we find their intersection points.

$2(1+\cos\theta) = 6\cos\theta \;\Rightarrow\; \cos\theta = 0.5, \quad \theta = \frac{\pi}{3}$ and $-\frac{\pi}{3}$

By symmetry

$$
\begin{aligned}
\text{Area} &= 2 \times 0.5\Big[\int_0^{\frac{\pi}{3}} (6\cos\theta)^2 d\theta - \int_0^{\frac{\pi}{3}} 4(1+\cos\theta)^2 d\theta\Big] \\
&= 36\int_0^{\frac{\pi}{3}} (\cos\theta)^2 d\theta - 4\int_0^{\frac{\pi}{3}} (1+\cos\theta)^2 d\theta \\
&= \int_0^{\frac{\pi}{3}} \Big[18\cos 2\theta + 18 - 4 - 8\cos\theta - 2\cos 2\theta - 2\Big] d\theta \\
&= (9\sin 2\theta + 18\theta - 4\theta - 8\sin\theta - \sin 2\theta - 2\theta)\Big|_0^{\frac{\pi}{3}} \\
&= 4\pi
\end{aligned}
$$

(c) The boundary of the region has two parts.

The first part is formed by the circle $r = 6\cos\theta \quad \frac{-\pi}{3} \leq \theta \leq \frac{\pi}{3}$

The length is $2 \times 3 \times \frac{2\pi}{3} = 4\pi$

The second part is formed by the cardioid. Using the formula $ds = \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$

$$
\begin{aligned}
ds &= \sqrt{(2+2\cos\theta)^2 + (-2\sin\theta)^2} d\theta \\
&= \sqrt{4 + 8\cos\theta + 4(\cos\theta)^2 + (4\sin\theta)^2} d\theta \\
&= \sqrt{8 + 8\cos\theta} d\theta
\end{aligned}
$$

The length is

$$
\begin{aligned}
2\int_0^{\frac{\pi}{3}} \sqrt{8+8\cos\theta} d\theta &= 2\int_0^{\frac{\pi}{3}} 4\cos\frac{\theta}{2} d\theta \\
&= 16\sin\frac{\theta}{2}\Big|_0^{\frac{\pi}{3}} = 8
\end{aligned}
$$

So the total length is $8 + 4\pi$