98學年度第1學期 微積分甲一組期末考解答

1. (16%) (a) Evaluate $I_1 = \int \frac{e^x + 1}{e^{2x} - 9}\, dx$.

   (b) Evaluate $I_2 = \int \frac{\sec^3 x}{\tan x}\, dx$.

   Sol:

   (a) Let $u = e^x$, and we have

$$\begin{aligned}
\int \frac{e^x + 1}{e^{2x} - 9}\, dx &= \int \frac{u+1}{u^2 - 9} \frac{du}{u} \\
&= \int -\frac{1}{9}\frac{1}{u} + \frac{2}{9}\frac{1}{u-3} - \frac{1}{9}\frac{1}{u+3}\, du \\
&= -\frac{1}{9}x + \frac{2}{9}\ln|e^x - 3| - \frac{1}{9}\ln|e^x + 3| + C
\end{aligned}$$

   (b)

$$\begin{aligned}
\int \frac{\sec^3 x}{\tan x}\, dx &= \int \frac{dx}{\sin x \cos^2 x} \\
&= \int \frac{\cos^2 x + \sin^2 x}{\sin x \cos^2 x}\, dx \\
&= \int \csc x\, dx + \int \frac{\sin x}{\cos^2 x}\, dx \\
&= -\ln|\csc x + \cot x| + \sec x + C
\end{aligned}$$

2. (8%) Find the integers A, B, and C such that $\int_2^4 (\ln x)^2\, dx = A(\ln 2)^2 + B\ln 2 + C$.

   Sol:

$$\begin{aligned}
\int_2^4 (\ln(x))^2 dx &= x(\ln x)^2\Big|_2^4 - 2\int_2^4 \ln x dx \\
&= x(\ln x)^2\Big|_2^4 - 2(x\ln x - x)\Big|_2^4 \\
&= (x\ln(x)^2 - 2x\ln(x) + 2x)\Big|_2^4 \\
&= 4\ln(4)^2 - 8\ln(4) + 8 - (2\ln(2)^2 - 4\ln(2) + 4) \\
&= 4\ln(4)^2 - 8\ln(4) - 2\ln(2)^2 + 4\ln(2) + 4 \\
&= 14\ln(2)^2 - 12\ln(2) + 4
\end{aligned}$$

3. (10%) Find the length of the curve $y = \sqrt{x-1}$ from $x = 1$ to $x = \frac{5}{4}$.

Sol:

From $(1,0)$ to $(\frac{5}{4}, \frac{1}{2})$, $x = y^2 + 1$.

$$
\begin{aligned}
\text{Its length} &= \int_0^{\frac{1}{2}} \sqrt{1 + (\frac{dx}{dy})^2}\, dy \\
&= \int_0^{\frac{1}{2}} \sqrt{1 + 4y^2}\, dy \quad (\text{let } y = \frac{1}{2}\tan\theta) \\
&= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^3\theta d\theta
\end{aligned}
$$

By

$$
\begin{aligned}
\int \sec^3\theta d\theta &= \int \sec\theta(\tan\theta)' d\theta = \sec\theta\tan\theta - \int (\sec\theta)'\tan\theta d\theta \\
&= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta = \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta \\
\Longrightarrow \int \sec^3\theta d\theta &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta d\theta \\
&= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + c,
\end{aligned}
$$

so

$$
\begin{aligned}
\text{the length} &= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^3\theta d\theta \\
&= \frac{1}{2}(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|)\Big|_0^{\frac{\pi}{4}} \\
&= \frac{1}{4}\sqrt{2} + \frac{1}{4}\ln(\sqrt{2}+1).
\end{aligned}
$$

4. (16%) Evaluate each integral, or show that the integral diverges.

   (a) $\displaystyle\int_{\frac{1}{2}}^{1} \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}\, dx.$

   (b) $\displaystyle\int_0^{\infty} \frac{1+x^2}{1+x^4}\, dx.$ (Hint. A particular substitution can be applied by observing that $\displaystyle\frac{1+x^2}{1+x^4} = \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}.$)

Sol:

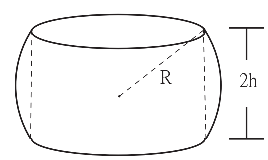
(a) Let $\sqrt{x} = u$, $x = u^2$, $dx = 2udu$

$$\lim_{k\to 1^-} \int_{\frac{1}{\sqrt{2}}}^{k} \frac{2\arcsin u du}{\sqrt{(1-u^2)}} = \lim_{k\to 1^-} (\arcsin(u))^2 \Big|_{\frac{1}{\sqrt{2}}}^{k}$$
$$= \lim_{k\to 1^-} (\arcsin(k))^2 - (\frac{\pi}{4})^2$$
$$= (\frac{\pi}{2})^2 - (\frac{\pi}{4})^2$$
$$= \frac{3}{16}\pi^2$$

(b) Let $x - \frac{1}{x} = u$, $du = (1 + \frac{1}{x^2})dx$

$$\lim_{m\to\infty} \int_{-m}^{0} \frac{du}{u^2+2} + \lim_{n\to\infty} \int_{0}^{n} \frac{du}{u^2+2} = \lim_{m\to\infty} (\frac{1}{\sqrt{2}} \arctan(\frac{u}{\sqrt{2}}))\Big|_{-m}^{0} + \lim_{n\to\infty} (\frac{1}{\sqrt{2}} \arctan(\frac{u}{\sqrt{2}}))\Big|_{0}^{n}$$
$$= \frac{1}{\sqrt{2}}\frac{\pi}{2} + \frac{1}{\sqrt{2}}\frac{\pi}{2}$$
$$= \frac{\pi}{2} = \frac{\sqrt{2}}{2}\pi$$

5. (16%) A ring with height $2h$ is made by drilling a hole through a ball with radius $R > h$.



(a) Find the total area of the inner and the outer surface of the ring.

(b) Find the value of h such that the volume of the ring is half of the volume of the whole ball.

Sol:

(a) The area of the inner surface $= 2\pi\sqrt{R^2 - h^2} \cdot 2h = 4\pi h\sqrt{R^2 - h^2}$

There are two meathod to compute the area of the outer surface:

method 1 : Let $x = R\cos(\theta)$, $y = R\sin(\theta)$; $\frac{dx}{d\theta} = -R\sin(\theta)$, $\frac{dy}{d\theta} = R\cos(\theta)$;

hence $ds = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta$

$$\int_{-\arcsin(\frac{h}{R})}^{\arcsin(\frac{h}{R})} 2\pi(R\cos(\theta))\sqrt{R^2\sin^2\theta + R^2\cos^2\theta} d\theta = \int_{-\arcsin(\frac{h}{R})}^{\arcsin(\frac{h}{R})} 2\pi R^2 \cos\theta d\theta$$
$$= 4\pi Rh$$

method 2 : $x^2+y^2=R^2$, therefore $\dfrac{dx}{dy}=\dfrac{-y}{\sqrt{R^2-y^2}}$, and $ds=\sqrt{1+(\frac{dx}{dy})^2}dy$

$$\int_{-h}^{h}2\pi x ds=2\pi\sqrt{R^2-y^2}\sqrt{1+\frac{y^2}{R^2-y^2}}dy=4\pi Rh$$

Hence, the total area of the inner and outer surface of the ring $=4\pi h(R+\sqrt{R^2-h^2})$

(b) There are also two methods to obtain the volume V of the ring:

method 1 :

$$\begin{aligned} V &= 2\int_0^h(\sqrt{R^2-y^2})^2\pi dy-(\sqrt{R^2-h^2})^2\pi\cdot(2h) \\ &= 2\pi(R^2y-\frac{y^3}{3})|_0^h-2h(R^2-h^2)\pi \\ &= \frac{4}{3}h^3\pi \end{aligned}$$

method 2 :

$$\begin{aligned} V &= 2\int_{\sqrt{R^2-h^2}}^{R}2\pi x\sqrt{R^2-x^2}dx \\ &= -2\pi\int_{\sqrt{R^2-h^2}}^{R}\sqrt{R^2-x^2}d(R^2-x^2) \\ &= \frac{4}{3}h^3\pi \end{aligned}$$

Volume of the ring $=\dfrac{1}{2}$ Volume of the whole ball

$$\Rightarrow \frac{4}{3}h^3\pi=\frac{1}{2}\cdot\frac{4}{3}\pi R^3$$
$$\Rightarrow h=2^{-\frac{1}{3}}R$$

6. (14%) Let Γ_1 be the curve $(x^2+y^2)^2=a^2(x^2-y^2)$, and Γ_2 be the curve $x^2+y^2=\dfrac{a^2}{2}$, $a>0$.

(a) Find all points of intersection of Γ_1 and Γ_2.

(b) Find the area of the region that lies inside Γ_1 and Γ_2. (Hint. Use polar coordinates.)

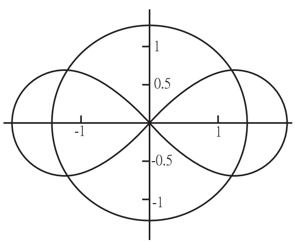
Sol:

(a) Using polar coordinates to change both eqations to $r^2=a^2\cos 2\theta$ and $r=\dfrac{a}{\sqrt{2}}$.

After solving $(\frac{a}{\sqrt{2}})^2 = r^2 = a^2\cos 2\theta$, we have $\cos 2\theta = \frac{1}{2}$ which implies $\theta = \pm\frac{\pi}{5},\ \pm\frac{5\pi}{6}$.
They intersect at the points
$(r,\theta) = (\frac{a}{\sqrt{2}}, \frac{\pi}{6}), (\frac{a}{\sqrt{2}}, \frac{-\pi}{6}), (\frac{a}{\sqrt{2}}, \frac{5\pi}{6}), (\frac{a}{\sqrt{2}}, \frac{-5\pi}{6})$.

(b) By symmetry, the area is equal to

$$\begin{aligned}
&4\Big[\int_0^{\frac{\pi}{6}} \frac{1}{2}(\frac{a}{\sqrt{2}})^2 d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2}a^2\cos 2\theta d\theta\Big] \\
&= 4\Big[\frac{\pi a^2}{24} + \frac{a^2}{4}\sin 2\theta\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}\Big] \\
&= a^2(1 - \frac{\sqrt{3}}{2} + \frac{\pi}{6}).
\end{aligned}$$



7. (10%) Solve the differential equation $y' - (\sec x)y = (\cos^2 x)y^2$ with initial condition $y(0) = 1$.

Sol:

Let $u = y^{1-2} = y^{-1}$, $u' = -y^{-2}y'$

$\frac{y'}{-y^2} - (\sec x)\frac{y}{-y^2} = (\cos^2 x)\frac{y^2}{-y^2}$

$u' + (\sec x)u = -(\cos^2 x)$

$e^{\int \sec x dx} = e^{\ln|\tan x + \sec x|} = |\tan x + \sec x| = \tan x + \sec x$

As x near 0^+, $y(0) = 1 > 0$, $\tan x + \sec x > 0$

$(u(\sec x + \tan x))' = -\cos^2 x(\tan x + \sec x) = -\sin x\cos x - \cos x$

$u(\sec x + \tan x) = \frac{1}{2}\cos^2 x - \sin x + C$

$u(0) = 1, C = \frac{1}{2}$

$y(x) = \frac{1}{u(x)} = \frac{\tan x + \sec x}{\frac{1}{2} + \frac{1}{2}\cos^2 x - \sin x}$

8. (10%) (a) Find integer k such that $\lim_{x\to 0} \frac{x - \sin x}{x^k}$ exists and is non-zero.

(b) Apply Mean Value Theorem for Integrals and the result in (a) to determine the range of p such that $\lim_{x\to 0^+} \int_{\sin x}^{x} t^p f(t)\, dt$ exists, where $f(t)$ is a continuous function in t and $f(0) \neq 0$.

Sol:

(a)

$$\lim_{x\to 0} \frac{x - \sin x}{x^k} = \lim_{x\to 0} \frac{1-\cos x}{kx^{k-1}}$$
$$= \lim_{x\to 0} \frac{\sin x}{k(k-1)x^{k-2}} = \lim_{x\to 0} \frac{\cos x}{k(k-1)(k-2)x^{k-3}}$$

$\because \cos 0 = 1, \quad \therefore k-3=0,\ k=3$

$\lim_{x\to 0} \frac{x-\sin x}{x^k} = \frac{1}{6} \neq 0$

If $k<3$ then $\lim_{x\to 0} \frac{x-\sin x}{x^k} = 0$ If $k>3$ then $\lim_{x\to 0} \frac{x-\sin x}{x^k} = \infty$

(b) Mean value theorem of integral:

$\int_{\sin x}^{x} t^p f(t) dt = c^p f(c)(x-\sin x)$ for some $c \in (\sin x, x)$

Note that $x > \sin x$ for $x > 0$

$\lim_{x\to 0} \int_{\sin x}^{x} t^p f(t) dt = \lim_{x\to 0} c^p f(c)(x-\sin x) = \lim_{x\to 0} c^{p+3}(\frac{x}{c})^3(\frac{x-\sin x}{x^3})$

$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \frac{1}{6}$

$1 = \lim_{x\to 0}(\frac{x}{x})^3 \leq \lim_{x\to 0}(\frac{x}{c})^3 \leq \lim_{x\to 0}(\frac{x}{\sin x})^3 = 1$

If $\lim_{x\to 0} c^{p+3}$ exists, then the limit exists $\therefore p+3 \geq 0 \therefore p \geq -3$ the limit exists