# Section 7.3 Trigonometric Substitution

20. Evaluate the integral. $\int_0^1 \frac{dx}{(x^2+1)^2}$

**Solution:**

Let $x = \tan\theta$, so $dx = \sec^2\theta\, d\theta$, $x = 0 \quad\Rightarrow\quad \theta = 0$, and $x = 1 \quad\Rightarrow\quad \theta = \frac{\pi}{4}$. Then

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^2}$$
$$= \int_0^{\pi/4} \cos^2\theta\, d\theta = \int_0^{\pi/4} \frac{1}{2}\left(1+\cos 2\theta\right)\, d\theta$$
$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{4}+\tfrac{1}{2}\right) - 0\right] = \tfrac{\pi}{8} + \tfrac{1}{4}$$

30. Evaluate the integral. $\int_0^1 \sqrt{x - x^2}dx$

**Solution:**

$$\int_0^1 \sqrt{x-x^2}\, dx = \int_0^1 \sqrt{\tfrac{1}{4} - \left(x^2 - x + \tfrac{1}{4}\right)}\, dx = \int_0^1 \sqrt{\tfrac{1}{4} - \left(x - \tfrac{1}{2}\right)^2}\, dx$$
$$= \int_{-\pi/2}^{\pi/2} \sqrt{\tfrac{1}{4} - \tfrac{1}{4}\sin^2\theta}\; \tfrac{1}{2}\cos\theta\, d\theta \qquad \begin{bmatrix} x - \frac{1}{2} = \frac{1}{2}\sin\theta, \\ dx = \frac{1}{2}\cos\theta\, d\theta \end{bmatrix}$$
$$= 2\int_0^{\pi/2} \tfrac{1}{2}\cos\theta\, \tfrac{1}{2}\cos\theta\, d\theta = \tfrac{1}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta = \tfrac{1}{2}\int_0^{\pi/2} \tfrac{1}{2}(1+\cos 2\theta)\, d\theta$$
$$= \tfrac{1}{4}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{1}{4}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{8}$$

39. Find the average value of $f(x) = \sqrt{x^2-1}/x$, $1 \le x \le 7$.

**Solution:**

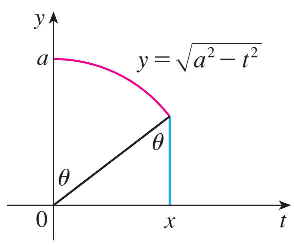
The average value of $f(x) = \sqrt{x^2-1}/x$ on the interval $[1, 7]$ is

$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\, dx = \frac{1}{6}\int_0^\alpha \frac{\tan\theta}{\sec\theta}\cdot \sec\theta\tan\theta\, d\theta \qquad \begin{bmatrix} \text{where } x = \sec\theta,\, dx = \sec\theta\tan\theta\, d\theta, \\ \sqrt{x^2-1} = \tan\theta, \text{ and } \alpha = \sec^{-1} 7 \end{bmatrix}$$
$$= \tfrac{1}{6}\int_0^\alpha \tan^2\theta\, d\theta = \tfrac{1}{6}\int_0^\alpha (\sec^2\theta - 1)\, d\theta = \tfrac{1}{6}\left[\tan\theta - \theta\right]_0^\alpha$$
$$= \tfrac{1}{6}(\tan\alpha - \alpha) = \tfrac{1}{6}\left(\sqrt{48} - \sec^{-1} 7\right)$$

45. (a) Use trigonometric substitution to verify that

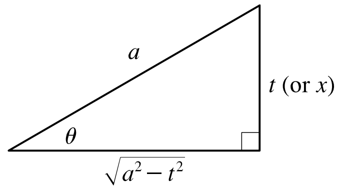
$$\int_0^x \sqrt{a^2-t^2}dt = \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2}$$

(b) Use the figure to give trigonometric interpretations of both terms on the right side of the equation in part (a).



**Solution:**

(a) Let $t = a\sin\theta$, $dt = a\cos\theta\,d\theta$, $t = 0 \quad\Rightarrow\quad \theta = 0$ and $t = x \quad\Rightarrow$ $\theta = \sin^{-1}(x/a)$. Then




$$\int_0^x \sqrt{a^2 - t^2}\,dt = \int_0^{\sin^{-1}(x/a)} a\cos\theta\,(a\cos\theta\,d\theta) = a^2 \int_0^{\sin^{-1}(x/a)} \cos^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\sin^{-1}(x/a)} (1+\cos 2\theta)\,d\theta = \frac{a^2}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\sin^{-1}(x/a)} = \frac{a^2}{2}\left[\theta + \sin\theta\,\cos\theta\right]_0^{\sin^{-1}(x/a)}$$

$$= \frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right) - 0\right] = \tfrac{1}{2}a^2\sin^{-1}(x/a) + \tfrac{1}{2}x\sqrt{a^2-x^2}$$

(b) The integral $\int_0^x \sqrt{a^2-t^2}\,dt$ represents the area under the curve $y = \sqrt{a^2-t^2}$ between the vertical lines $t = 0$ and $t = x$. The figure shows that this area consists of a triangular region and a sector of the circle $t^2 + y^2 = a^2$. The triangular region has base $x$ and height $\sqrt{a^2-x^2}$, so its area is $\frac{1}{2}x\sqrt{a^2-x^2}$. The sector has area $\frac{1}{2}a^2\theta = \frac{1}{2}a^2\sin^{-1}(x/a)$.

47. A torus is generated by rotating the circle $x^2 + (y-R)^2 = r^2$ about the $x$-axis. Find the volume enclosed by the torus.

**Solution:**

We use cylindrical shells and assume that $R > r$. $x^2 = r^2 - (y-R)^2 \quad\Rightarrow\quad x = \pm\sqrt{r^2-(y-R)^2}$, so $g(y) = 2\sqrt{r^2-(y-R)^2}$ and

$$V = \int_{R-r}^{R+r} 2\pi y\cdot 2\sqrt{r^2-(y-R)^2}\,dy = \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \qquad [\text{where } u = y - R]$$

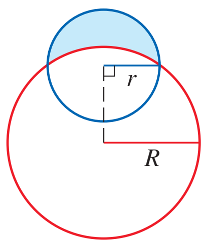
$$= 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du + 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du \qquad \left[\begin{array}{c}\text{where } u = r\sin\theta\,,\ du = r\cos\theta\,d\theta\\ \text{in the second integral}\end{array}\right]$$

$$= 4\pi\left[-\tfrac{1}{3}(r^2-u^2)^{3/2}\right]_{-r}^{r} + 4\pi R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta = -\tfrac{4\pi}{3}(0-0) + 4\pi R r^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$$= 2\pi R r^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta = 2\pi R r^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2\pi^2 R r^2$$

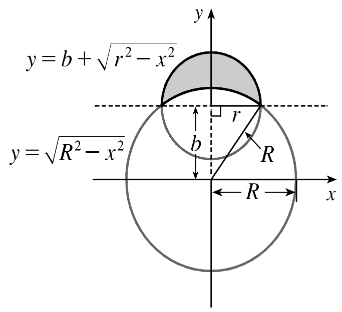
*Another method:* Use washers instead of shells, so $V = 8\pi R\int_0^r \sqrt{r^2-y^2}\,dy$ as in Exercise 6.2.63(a), but evaluate the integral using $y = r\sin\theta$.

49. Find the area of the crescent-shaped region (called a lune) bounded by arcs of circles with radii $r$ and $R$. (See the figure.)




**Solution:**

Let the equation of the large circle be $x^2 + y^2 = R^2$. Then the equation of the small circle is $x^2 + (y - b)^2 = r^2$, where $b = \sqrt{R^2 - r^2}$ is the distance between the centers of the circles. The desired area is




$$A = \int_{-r}^{r} \left[\left(b + \sqrt{r^2 - x^2}\,\right) - \sqrt{R^2 - x^2}\,\right] dx$$
$$= 2\int_0^r \left(b + \sqrt{r^2 - x^2} - \sqrt{R^2 - x^2}\,\right) dx$$
$$= 2\int_0^r b\,dx + 2\int_0^r \sqrt{r^2 - x^2}\,dx - 2\int_0^r \sqrt{R^2 - x^2}\,dx$$

The first integral is just $2br = 2r\sqrt{R^2 - r^2}$. The second integral represents the area of a quarter-circle of radius $r$, so its value is $\frac{1}{4}\pi r^2$. To evaluate the other integral, note that

$$\int \sqrt{a^2 - x^2}\,dx = \int a^2\cos^2\theta\,d\theta \quad [x = a\sin\theta,\, dx = a\cos\theta\,d\theta] \quad = \left(\tfrac{1}{2}a^2\right)\int(1 + \cos 2\theta)\,d\theta$$
$$= \tfrac{1}{2}a^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{2}a^2(\theta + \sin\theta\,\cos\theta) + C$$
$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2 - x^2}}{a} + C = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

Thus, the desired area is

$$A = 2r\sqrt{R^2 - r^2} + 2\left(\tfrac{1}{4}\pi r^2\right) - \left[R^2\arcsin(x/R) + x\sqrt{R^2 - x^2}\,\right]_0^r$$
$$= 2r\sqrt{R^2 - r^2} + \tfrac{1}{2}\pi r^2 - \left[R^2\arcsin(r/R) + r\sqrt{R^2 - r^2}\,\right] = r\sqrt{R^2 - r^2} + \tfrac{\pi}{2}r^2 - R^2\arcsin(r/R)$$