Section 7.2 Trigonometric Integrals

14. Evaluate the integral $\int(1+\sqrt[3]{\sin t})\cos^3 t dt$.

Solution:

$$\int\left(1+\sqrt[3]{\sin t}\right)\cos^3 t\,dt = \int\left(1+(\sin t)^{1/3}\right)\cos^2 t\cos t\,dt = \int\left(1+(\sin t)^{1/3}\right)(1-\sin^2 t)\cos t\,dt$$

$$\overset{s}{=}\int(1+u^{1/3})(1-u^2)\,du = \int(1-u^2+u^{1/3}-u^{7/3})\,du$$

$$= u-\tfrac{1}{3}u^3+\tfrac{3}{4}u^{4/3}-\tfrac{3}{10}u^{10/3}+C$$

$$= \sin t-\tfrac{1}{3}\sin^3 t+\tfrac{3}{4}\sqrt[3]{\sin^4 t}-\tfrac{3}{10}\sqrt[3]{\sin^{10} t}+C$$

30. Evaluate the integral. $\int_0^{\pi/4}\tan^4 t dt$

Solution:

Let $u = xe^x, dv = \dfrac{1}{(1+x)^2}\,dx \quad\Rightarrow\quad du = (xe^x+e^x)\,dx = e^x(x+1)\,dx,\ v = -\dfrac{1}{1+x}$. By (6),

$$\int_0^1\frac{xe^x}{(1+x)^2}\,dx = \left[-\frac{xe^x}{1+x}\right]_0^1 - \int_0^1\left(-\frac{1}{1+x}\right)e^x(1+x)\,dx = \left(-\frac{e}{2}+0\right)+\int_0^1 e^x\,dx = -\frac{1}{2}e+\left[e^x\right]_0^1$$

$$= -\tfrac{1}{2}e+e-1 = \tfrac{1}{2}e-1$$

32. Evaluate the integral. $\int\tan^2 x\sec x dx$

Solution:

$$\int\tan^2 x\ \sec x\,dx = \int(\sec^2 x-1)\ \sec x\,dx = \int\sec^3 x\,dx - \int\sec x\,dx$$

$$= \tfrac{1}{2}(\sec x\ \tan x+\ln|\sec x+\tan x|)-\ln|\sec x+\tan x|+C \qquad \text{[by Example 8 and (1)]}$$

$$= \tfrac{1}{2}(\sec x\ \tan x-\ln|\sec x+\tan x|)+C$$

56. Evaluate the integral. $\int\frac{1}{\sec\theta+1}d\theta$

Solution:

$$\int\frac{1}{\sec\theta+1}\,d\theta = \int\frac{1}{\sec\theta+1}\cdot\frac{\sec\theta-1}{\sec\theta-1}\,d\theta = \int\frac{\sec\theta-1}{\sec^2\theta-1}\,d\theta = \int\frac{\sec\theta-1}{\tan^2\theta}\,d\theta$$

$$= \int\frac{\cos\theta}{\sin^2\theta}\,d\theta - \int\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int\frac{\cos\theta\,d\theta}{\sin^2\theta} - \int\frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta \overset{s}{=} \int\frac{1}{u^2}\,du - \int\csc^2\theta\,d\theta + \int d\theta$$

$$= -\frac{1}{\sin\theta}+\cot\theta+\theta+C$$

Alternate solution:

$$\int\frac{1}{\sec\theta+1}\,d\theta = \int\frac{\cos\theta}{1+\cos\theta}\,d\theta = \int\frac{2\cos^2\left(\frac{\theta}{2}\right)-1}{2\cos^2\left(\frac{\theta}{2}\right)}\,d\theta \qquad \text{[double-angle identities]}$$

$$= \int 1\,d\theta - \int\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\,d\theta = \theta-\tan\left(\frac{\theta}{2}\right)+C$$

63. Find the average value of the function $f(x) = \sin^2 x\cos^3 x$ on the interval $[-\pi,\pi]$.

Solution:

$$f_{\text{ave}} = \tfrac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 x\ \cos^3 x\,dx = \tfrac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 x\,(1-\sin^2 x)\ \cos x\,dx$$

$$= \tfrac{1}{2\pi}\int_0^0 u^2(1-u^2)\,du\ \ [\text{where } u=\sin x]\quad = 0$$

71. Find the volume obtained by rotating the region bounded by the curves about the given axis.

$$y = \sin x, \quad y = \cos x, \quad 0 \le x \le \frac{\pi}{4}; \quad \text{about } y = 1$$

Solution:

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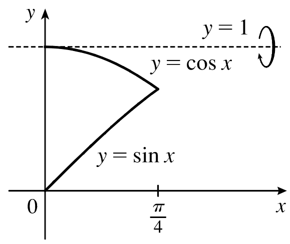

$$\begin{aligned} V &= \int_0^{\pi/4} \pi\left[(1-\sin x)^2 - (1-\cos x)^2\right] dx \\ &= \pi \int_0^{\pi/4} \left[(1 - 2\sin x + \sin^2 x) - (1 - 2\cos x + \cos^2 x)\right] dx \\ &= \pi \int_0^{\pi/4} (2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\, dx \\ &= \pi \int_0^{\pi/4} (2\cos x - 2\sin x - \cos 2x)\, dx = \pi\left[2\sin x + 2\cos x - \tfrac{1}{2}\sin 2x\right]_0^{\pi/4} \\ &= \pi\left[\left(\sqrt{2} + \sqrt{2} - \tfrac{1}{2}\right) - (0 + 2 - 0)\right] = \pi\left(2\sqrt{2} - \tfrac{5}{2}\right) \end{aligned}$$