Section 6.5 Average Value of a Function

13. If f is continuous and $\int_1^3 f(x)dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.

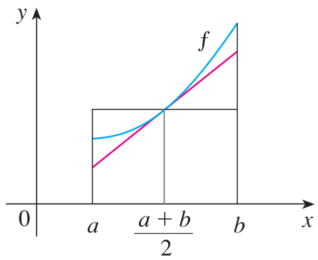
Solution:

f is continuous on $[1, 3]$, so by the Mean Value Theorem for Integrals there exists a number c in $[1, 3]$ such that

$\int_1^3 f(x)\,dx = f(c)(3-1) \quad \Rightarrow \quad 8 = 2f(c)$; that is, there is a number c such that $f(c) = \frac{8}{2} = 4$.

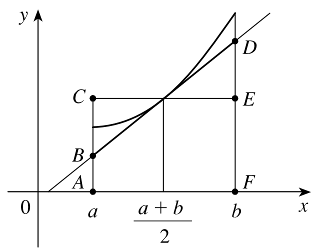
25. Use the diagram to show that if f is concave upward on $[a, b]$, then

$$f_{\text{ave}} > f\left(\frac{a+b}{2}\right)$$



Solution:

$$\begin{aligned} f_{\text{ave}} &= \tfrac{1}{b-a}\textstyle\int_a^b f(x)\,dx \\ &> \tfrac{1}{b-a}\ \text{(area of trapezoid } ABDF\text{)} \\ &= \tfrac{1}{b-a}\ \text{(area of rectangle } ACEF\text{)} \\ &= \tfrac{1}{b-a}\left[f\left(\tfrac{a+b}{2}\right)\cdot(b-a)\right] \\ &= f\left(\tfrac{a+b}{2}\right) \end{aligned}$$



26. Let $f_{\text{ave}}[a, b]$ denote the average value of f on the interval $[a, b]$. Show that if $a < c < b$, then

$$f_{\text{ave}}[a, b] = \left(\frac{c-a}{b-a}\right) f_{\text{ave}}[a, c] + \left(\frac{b-c}{b-a}\right) f_{\text{ave}}[c, b]$$

Solution:

$$\begin{aligned} f_{\text{ave}}\,[a, b] &= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^c f(x)\,dx + \frac{1}{b-a}\int_c^b f(x)\,dx \\ &= \frac{c-a}{b-a}\left[\frac{1}{c-a}\int_a^c f(x)\,dx\right] + \frac{b-c}{b-a}\left[\frac{1}{b-c}\int_c^b f(x)\,dx\right] = \frac{c-a}{b-a}\,f_{\text{ave}}\,[a, c] + \frac{b-c}{b-a}\,f_{\text{ave}}\,[c, b] \end{aligned}$$