Section 4.4 Indeterminate Forms and l'Hospital's Rule

60. Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$$

Solution:

$y=\left(1+\frac{a}{x}\right)^{bx} \quad\Rightarrow\quad \ln y = bx\ln\left(1+\frac{a}{x}\right)$, so

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{b\ln(1+a/x)}{1/x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{b\left(\dfrac{1}{1+a/x}\right)\left(-\dfrac{a}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{ab}{1+a/x}=ab \quad\Rightarrow$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=\lim_{x\to\infty}e^{\ln y}=e^{ab}.$$

69. Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

$$\lim_{x\to 0^+}\frac{x^x-1}{\ln x+x-1}$$

Solution:

The given limit is $\lim\limits_{x\to 0^+}\dfrac{x^x-1}{\ln x+x-1}$. Note that $y=x^x \quad\Rightarrow\quad \ln y = x\ln x$, so

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\dfrac{1}{x}}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}}=\lim_{x\to 0^+}(-x)=0 \quad\Rightarrow\quad \lim_{x\to 0^+}x^x=\lim_{x\to 0^+}e^{\ln y}=e^0=1.$$

Therefore, the numerator of the given limit has limit $1-1=0$ as $x\to 0^+$. The denominator of the given limit $\to -\infty$ as $x\to 0^+$ since $\ln x\to-\infty$ as $x\to 0^+$. Thus, $\lim\limits_{x\to 0^+}\dfrac{x^x-1}{\ln x+x-1}=0$.

70. Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

$$\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$$

Solution:

$$y=\left(\frac{2x-3}{2x+5}\right)^{2x+1} \quad\Rightarrow\quad \ln y=(2x+1)\,\ln\left(\frac{2x-3}{2x+5}\right) \quad\Rightarrow$$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{2/(2x-3)-2/(2x+5)}{-2/(2x+1)^2}=\lim_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$=\lim_{x\to\infty}\frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)}=-8 \quad\Rightarrow\quad \lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}=\lim_{x\to\infty}e^{\ln y}=e^{-8}$$

76. Prove that

$$\lim_{x\to\infty}\frac{\ln x}{x^p}=0$$

for any number $p>0$. This shows that the logarithmic function approaches infinity more slowly than any power of x.

Solution:

This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^p}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1/x}{px^{p-1}}=\lim\limits_{x\to\infty}\dfrac{1}{px^p}=0$ since $p>0$.

78. What happens if you try to use l'Hospital's Rule to find the limit? Evaluate the limit using another method.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x}$$

Solution:

$\displaystyle\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^-} \frac{\sec x\,\tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x}$. Repeated applications of l'Hospital's Rule result in the original limit or the limit of the reciprocal of the function. Another method is to simplify first:

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x} = \lim_{x\to(\pi/2)^-} \frac{1/\cos x}{\sin x/\cos x} = \lim_{x\to(\pi/2)^-} \frac{1}{\sin x} = \frac{1}{1} = 1$$

90. For what values of a and b is the following equation true?

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0$$

Solution:

$\displaystyle L = \lim_{x\to 0}\left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = \lim_{x\to 0}\frac{\sin 2x + ax^3 + bx}{x^3} \overset{\text{H}}{=} \lim_{x\to 0}\frac{2\cos 2x + 3ax^2 + b}{3x^2}$. As $x \to 0$, $3x^2 \to 0$, and $(2\cos 2x + 3ax^2 + b) \to b + 2$, so the last limit exists only if $b + 2 = 0$, that is, $b = -2$. Thus,

$\displaystyle\lim_{x\to 0}\frac{2\cos 2x + 3ax^2 - 2}{3x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-4\sin 2x + 6ax}{6x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-8\cos 2x + 6a}{6} = \frac{6a-8}{6}$, which is equal to 0 if and only if $a = \frac{4}{3}$. Hence, $L = 0$ if and only if $b = -2$ and $a = \frac{4}{3}$.