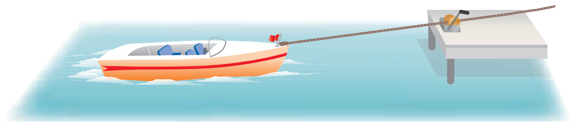
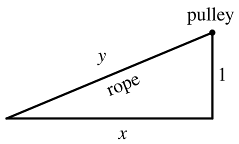
Section 3.9 Related Rates

22. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?



Solution:



Given $\frac{dy}{dt} = -1$ m/s, find $\frac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \quad \Rightarrow \quad 2y\frac{dy}{dt} = 2x\frac{dx}{dt} \quad \Rightarrow$ $\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt} = -\frac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so $\frac{dx}{dt} = -\frac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\frac{\sqrt{65}}{8} \approx 1.01$ m/s.

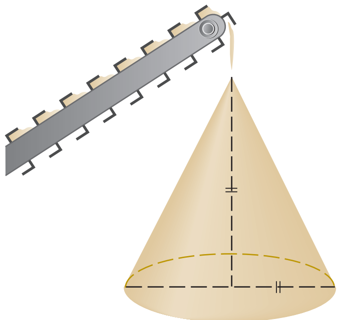
23. Use the fact that the distance(in meters) a dropped stone falls after t seconds is $d = 4.9t^2$. A woman stands near the edge of a cliff and drops a stone over the edge. Exactly one second later she drops another stone. One second after that, how fast is the distance between the two stones changing?

Solution:

Let x be the distance (in meters) the first dropped stone has traveled, and let y be the distance (in meters) the stone dropped one second later has traveled. Let t be the time (in seconds) since the woman drops the second stone. Using $d = 4.9t^2$, we have $x = 4.9(t+1)^2$ and $y = 4.9t^2$. Let z be the distance between the stones. Then $z = x - y$ and we have

$$\frac{dz}{dt} = \frac{dx}{dt} - \frac{dy}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = 9.8(t+1) - 9.8t = 9.8 \text{ m/s}.$$

29. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

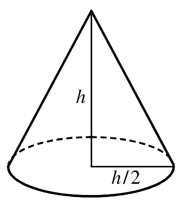


Solution:

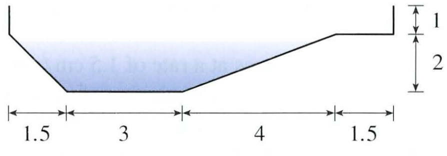
We are given that $\frac{dV}{dt} = 3 \text{ m}^3/\text{min}$. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12} \quad \Rightarrow$

$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \quad \Rightarrow \quad 3 = \frac{\pi h^2}{4}\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{12}{\pi h^2}$. When $h = 3$ m,

$\frac{dh}{dt} = \frac{12}{3^2\pi} = \frac{4}{3\pi} \approx 0.42$ m/min.



30. A swimming pool is 5m wide, 10m long, 1m deep at the shallow end, and 3m deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of $0.1\text{m}^3/\text{min}$, how fast is the water level rising when the depth at the deepest point is 1m?

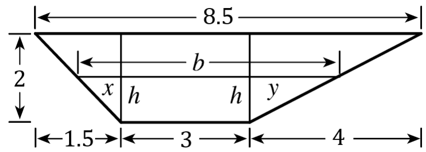


Solution:

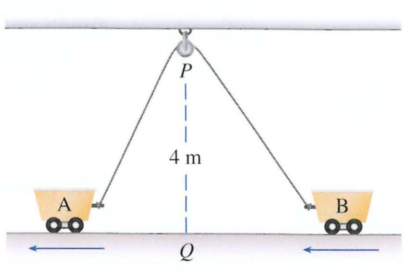
The figure is drawn without the top 1 meter.

$V = \frac{1}{2}(b+3)h(5) = \frac{5}{2}(b+3)h$ and, from similar triangles, $\frac{x}{h} = \frac{1.5}{2} = \frac{3}{4}$ and $\frac{y}{h} = \frac{4}{2} = 2$, so $b = x + 3 + y = \frac{3h}{4} + 3 + 2h = 3 + \frac{11h}{4}$. Thus, $V = \frac{5}{2}\left(6 + \frac{11h}{4}\right)h = 15h + \frac{55}{8}h^2$ and so $0.1 = \frac{dV}{dt} = \left(15 + \frac{55}{4}h\right)\frac{dh}{dt}$.

When $h = 1$, $\frac{dh}{dt} = \frac{0.1}{\left(15 + \frac{55}{4}\right)} = \frac{2}{575} \approx 0.00348$ m/min.



44. Two carts, A and B, are connected by a rope 12m long that passes over a pulley P.(See the figure.) The point Q is on the floor 4m directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 0.5m/s. How fast is cart B moving toward Q at the instant when cart A is 3m from Q?

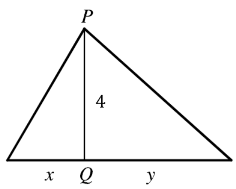


Solution:

Using Q for the origin, we are given $\frac{dx}{dt} = -0.5$ m/s and need to find $\frac{dy}{dt}$ when $x = -5$.

Using the Pythagorean Theorem twice, we have $\sqrt{x^2 + 4^2} + \sqrt{y^2 + 4^2} = 12$, the total length of the rope. Differentiating with respect to t, we get

$\frac{x}{\sqrt{x^2+4^2}}\frac{dx}{dt} + \frac{y}{\sqrt{y^2+4^2}}\frac{dy}{dt} = 0$, so $\frac{dy}{dt} = -\frac{x\sqrt{y^2+4^2}}{y\sqrt{x^2+4^2}}\frac{dx}{dt}$.



Now, when $x = -3$, $12 = \sqrt{(-3)^2 + 4^2} + \sqrt{y^2 + 4^2} = 5 + \sqrt{y^2 + 4^2} \Leftrightarrow \sqrt{y^2 + 4^2} = 7$, and $y = \sqrt{7^2 - 4^2} = \sqrt{33}$.

So when $x = -3$, $\frac{dy}{dt} = -\frac{(-3)(7)}{\sqrt{33}(5)}(-0.5) \approx -0.37$ m/s.

So cart B is moving towards Q at about 0.37 m/s.

53. Suppose that the volume V of a rolling snowball increases so that dV/dt is proportional to the surface area of the snowball at time t. Show that the radius r increases at a constant rate, that is, dr/dt is constant.

Solution:

The volume of the snowball is given by $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$. Since the volume is proportional to the surface area S, with $S = 4\pi r^2$, we also have $\frac{dV}{dt} = k \cdot 4\pi r^2$ for some constant k. Equating the two expressions for $\frac{dV}{dt}$ gives $4\pi r^2 \frac{dr}{dt} = k \cdot 4\pi r^2 \quad \Rightarrow \quad \frac{dr}{dt} = k$, that is, dr/dt is constant.