Section 3.6 Derivatives of Logarithmic and Inverse Trigonometric Functions

36. Differentiate f and find the domain of f.

$$f(x) = \ln \ln \ln x$$

Solution:

$f(x) = \ln \ln \ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{\ln \ln x} \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x}.$

$\mathrm{Dom}(f) = \{x \mid \ln \ln x > 0\} = \{x \mid \ln x > 1\} = \{x \mid x > e\} = (e, \infty).$

58. Find y' if $x^y = y^x$.

Solution:

$x^y = y^x \quad \Rightarrow \quad y \ln x = x \ln y \quad \Rightarrow \quad y \cdot \dfrac{1}{x} + (\ln x) \cdot y' = x \cdot \dfrac{1}{y} \cdot y' + \ln y \quad \Rightarrow \quad y' \ln x - \dfrac{x}{y} y' = \ln y - \dfrac{y}{x} \quad \Rightarrow$

$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

78. Find the derivative of the function. Simplify where possible. $y = \arctan \sqrt{\frac{1-x}{1+x}}$.

Solution:

$y = \arctan \sqrt{\dfrac{1-x}{1+x}} = \arctan \left(\dfrac{1-x}{1+x}\right)^{1/2} \quad \Rightarrow$

$$y' = \frac{1}{1 + \left(\sqrt{\frac{1-x}{1+x}}\right)^2} \cdot \frac{d}{dx}\left(\frac{1-x}{1+x}\right)^{1/2} = \frac{1}{1 + \frac{1-x}{1+x}} \cdot \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \cdot \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2}$$

$$= \frac{1}{\frac{1+x}{1+x} + \frac{1-x}{1+x}} \cdot \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2} \cdot \frac{-2}{(1+x)^2} = \frac{1+x}{2} \cdot \frac{1}{2} \cdot \frac{(1+x)^{1/2}}{(1-x)^{1/2}} \cdot \frac{-2}{(1+x)^2}$$

$$= \frac{-1}{2(1-x)^{1/2}(1+x)^{1/2}} = \frac{-1}{2\sqrt{1-x^2}}$$

83. **Derivatives of Inverse Functions** Suppose that f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

Solution:

If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating implicitly with respect to x and remembering that y is a function of x,

we get $f'(y)\dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1}{f'(y)} \quad \Rightarrow \quad \left(f^{-1}\right)'(x) = \dfrac{1}{f'(f^{-1}(x))}.$

85. Use the formula in Exercise 83.

If $f(x) = x + e^x$, find $(f^{-1})'(1)$.

Solution:

$f(x) = x + e^x \quad \Rightarrow \quad f'(x) = 1 + e^x$. Observe that $f(0) = 1$, so that $f^{-1}(1) = 0$. By Exercise 83, we have

$(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{1 + e^0} = \dfrac{1}{1+1} = \dfrac{1}{2}.$