

Section 3.3 Derivatives of Trigonometric Functions

46. Find the limit. $\lim_{x \rightarrow 0} \frac{\sin x}{\sin \pi x}$.

Solution:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x}{\sin \pi x} &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\pi x}{\sin \pi x} \cdot \frac{1}{\pi} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} \cdot \frac{1}{\pi} \quad [\theta = \pi x] \\ &= 1 \cdot \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{\theta}} \cdot \frac{1}{\pi} = 1 \cdot 1 \cdot \frac{1}{\pi} = \frac{1}{\pi} \end{aligned}$$

60. Find the limit. $\lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2+x-2}$

Solution:

$$\lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2+x-2} = \lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x+2)(x-1)} = \lim_{x \rightarrow 1} \frac{1}{x+2} \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x-1} = \frac{1}{3} \cdot 1 = \frac{1}{3}$$

62. Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

$$\frac{d^{35}}{dx^{35}}(x \sin x).$$

Solution:

Let $f(x) = x \sin x$ and $h(x) = \sin x$, so $f(x) = xh(x)$. Then $f'(x) = h(x) + xh'(x)$,

$$f''(x) = h'(x) + h'(x) + xh''(x) = 2h'(x) + xh''(x),$$

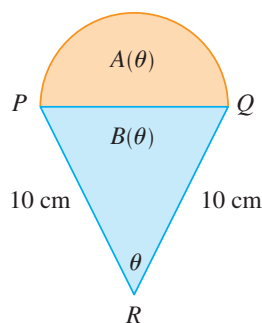
$$f'''(x) = 2h''(x) + h''(x) + xh'''(x) = 3h''(x) + xh'''(x), \dots, f^{(n)}(x) = nh^{(n-1)}(x) + xh^{(n)}(x).$$

Since $34 = 4(8) + 2$, we have $h^{(34)}(x) = h^{(2)}(x) = \frac{d^2}{dx^2}(\sin x) = -\sin x$ and $h^{(35)}(x) = -\cos x$.

Thus, $\frac{d^{35}}{dx^{35}}(x \sin x) = 35h^{(34)}(x) + xh^{(35)}(x) = -35 \sin x - x \cos x$.

66. A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like a two-dimensional ice-cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta \rightarrow 0^+} \frac{A(\theta)}{B(\theta)}.$$



Solution:

We get the following formulas for r and h in terms of θ :

$$\sin \frac{\theta}{2} = \frac{r}{10} \Rightarrow r = 10 \sin \frac{\theta}{2} \quad \text{and} \quad \cos \frac{\theta}{2} = \frac{h}{10} \Rightarrow h = 10 \cos \frac{\theta}{2}$$

Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\begin{aligned} \lim_{\theta \rightarrow 0^+} \frac{A(\theta)}{B(\theta)} &= \lim_{\theta \rightarrow 0^+} \frac{\frac{1}{2}\pi r^2}{rh} = \frac{1}{2}\pi \lim_{\theta \rightarrow 0^+} \frac{r}{h} = \frac{1}{2}\pi \lim_{\theta \rightarrow 0^+} \frac{10 \sin(\theta/2)}{10 \cos(\theta/2)} \\ &= \frac{1}{2}\pi \lim_{\theta \rightarrow 0^+} \tan(\theta/2) = 0 \end{aligned}$$

