Section 3.1 Derivatives of Polynomials and Exponential Functions

38. Find an equation of the tangent line to the curve at the given point.

$$y = 2e^x + x, \quad (0, 2)$$

Solution:

$y = 2e^x + x \quad \Rightarrow \quad y' = 2e^x + 1$. At $(0, 2)$, $y' = 2e^0 + 1 = 3$ and an equation of the tangent line is $y - 2 = 3(x - 0)$ or $y = 3x + 2$.

86. Find numbers a and b such that the given function g is differentiable at 1.

$$g(x) = \begin{cases} ax^3 - 3x & \text{if } x \leq 1 \\ bx^2 + 2 & \text{if } x > 1 \end{cases}$$

Solution:

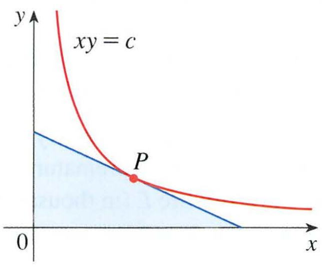
We have $g(x) = \begin{cases} ax^3 - 3x & \text{if} \quad x \leq 1 \\ bx^2 + 2 & \text{if} \quad x > 1 \end{cases}$

For $x < 1$, $g'(x) = a(3x^2) - 3(1) = 3ax^2 - 3$, so $g'_{-}(1) = 3a(1)^2 - 3 = 3a - 3$. For $x > 1$, $g'(x) = b(2x) + 0 = 2bx$, so $g'_{+}(1) = 2b(1) = 2b$. For g to be differentiable at $x = 1$, we need $g'_{-}(1) = g'_{+}(1)$, so $3a - 3 = 2b$, or $b = \dfrac{3a - 3}{2}$. For g to be continuous at $x = 1$, we need $g_{-}(1) = a - 3$ equal to $g_{+}(1) = b + 2$. So we have the system of two equations: $a - 3 = b + 2$, $b = \dfrac{3a - 3}{2}$. Substituting the second equation into the first equation we have $a - 3 = \dfrac{3a - 3}{2} + 2 \quad \Rightarrow$ $2a - 6 = 3a - 3 + 4 \quad \Rightarrow \quad a = -7$ and $b = \dfrac{3(-7) - 3}{2} = -12$.

88. A tangent line is drawn to the hyperbola $xy = c$ at a point P as shown in the figure.

(a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is P.

(b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.



Solution:

(a) $xy = c \quad \Rightarrow \quad y = \dfrac{c}{x}$. Let $P = \left(a, \dfrac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\dfrac{c}{a^2}$. Its equation is

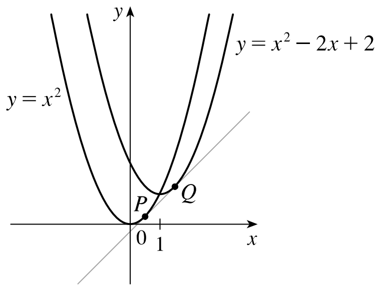
$y - \dfrac{c}{a} = -\dfrac{c}{a^2}(x - a)$ or $y = -\dfrac{c}{a^2}x + \dfrac{2c}{a}$, so its y-intercept is $\dfrac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x-intercept is $2a$.

The midpoint of the line segment joining $\left(0, \dfrac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \dfrac{c}{a}\right) = P$.

(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

90. Sketch the parabolas $y = x^2$ and $y = x^2 - 2x + 2$. Do you think there is a line that is tangent to both curves? If so, find its equation. If not, why not?

Solution:



From the sketch, it appears that there may be a line that is tangent to both curves. The slope of the line through the points $P(a, a^2)$ and $Q(b, b^2 - 2b + 2)$ is $\dfrac{b^2 - 2b + 2 - a^2}{b - a}$. The slope of the tangent line at P is $2a$ $[y' = 2x]$ and at Q is $2b - 2$ $[y' = 2x - 2]$. All three slopes are equal, so $2a = 2b - 2 \;\Leftrightarrow\; a = b - 1$.

Also, $2b - 2 = \dfrac{b^2 - 2b + 2 - a^2}{b - a} \;\Rightarrow\; 2b - 2 = \dfrac{b^2 - 2b + 2 - (b-1)^2}{b - (b-1)} \;\Rightarrow\; 2b - 2 = b^2 - 2b + 2 - b^2 + 2b - 1 \;\Rightarrow$

$2b = 3 \;\Rightarrow\; b = \frac{3}{2}$ and $a = \frac{3}{2} - 1 = \frac{1}{2}$. Thus, an equation of the tangent line at P is $y - \left(\frac{1}{2}\right)^2 = 2\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)$ or

$y = x - \frac{1}{4}$.