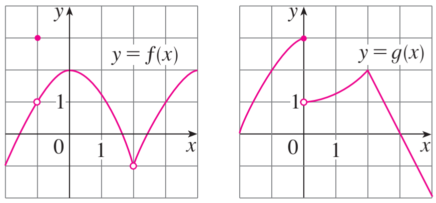
Section 2.3 Calculating Limits Using the Limit Laws

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x\to 2}[f(x)+g(x)]$ (b) $\lim_{x\to 0}[f(x)-g(x)]$ (c) $\lim_{x\to -1}[f(x)g(x)]$ (d) $\lim_{x\to 3}\frac{f(x)}{g(x)}$ (e) $\lim_{x\to 2}[x^2 f(x)]$ (f) $f(-1)+\lim_{x\to -1} g(x)$



Solution:

(a) $\lim_{x\to 2}[f(x)+g(x)] = \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x)$ [Limit Law 1]

$= -1+2$

$= 1$

(b) $\lim_{x\to 0} f(x)$ exists, but $\lim_{x\to 0} g(x)$ does not exist, so we cannot apply Limit Law 2 to $\lim_{x\to 0}[f(x)-g(x)]$.

The limit does not exist.

(c) $\lim_{x\to -1}[f(x)\,g(x)] = \lim_{x\to -1} f(x)\cdot \lim_{x\to -1} g(x)$ [Limit Law 4]

$= 1\cdot 2$

$= 2$

(d) $\lim_{x\to 3} f(x) = 1$, but $\lim_{x\to 3} g(x) = 0$, so we cannot apply Limit Law 5 to $\lim_{x\to 3}\dfrac{f(x)}{g(x)}$. The limit does not exist.

Note: $\lim_{x\to 3^-}\dfrac{f(x)}{g(x)} = \infty$ since $g(x)\to 0^+$ as $x\to 3^-$ and $\lim_{x\to 3^+}\dfrac{f(x)}{g(x)} = -\infty$ since $g(x)\to 0^-$ as $x\to 3^+$.

Therefore, the limit does not exist, even as an infinite limit.

(e) $\lim_{x\to 2}\left[x^2 f(x)\right] = \lim_{x\to 2} x^2\cdot\lim_{x\to 2} f(x)$ [Limit Law 4]

$= 2^2\cdot(-1)$

$= -4$

(f) $f(-1)+\lim_{x\to -1} g(x)$ is undefined since $f(-1)$ is not defined.

34. Evaluate the limit, if it exists. $\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

Solution:

$$\lim_{h\to 0}\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h} = \lim_{h\to 0}\frac{\dfrac{x^2-(x+h)^2}{(x+h)^2x^2}}{h} = \lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2} = \lim_{h\to 0}\frac{-h(2x+h)}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0}\frac{-(2x+h)}{x^2(x+h)^2} = \frac{-2x}{x^2\cdot x^2} = -\frac{2}{x^3}$$

42. Prove that $\lim_{x\to 0^+}\sqrt{x}e^{\sin(\pi/x)} = 0$.

Solution:

$-1\le \sin(\pi/x)\le 1 \;\Rightarrow\; e^{-1}\le e^{\sin(\pi/x)}\le e^1 \;\Rightarrow\; \sqrt{x}/e \le \sqrt{x}\,e^{\sin(\pi/x)} \le \sqrt{x}\,e$. Since $\lim_{x\to 0^+}(\sqrt{x}/e) = 0$ and $\lim_{x\to 0^+}(\sqrt{x}\,e) = 0$, we have $\lim_{x\to 0^+}\left[\sqrt{x}\,e^{\sin(\pi/x)}\right] = 0$ by the Squeeze Theorem.

54. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \le 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists. (i) $\lim_{x \to 1^-} g(x)$ (ii) $\lim_{x \to 1} g(x)$ (iii) $g(1)$ (iv) $\lim_{x \to 2^-} g(x)$ (v) $\lim_{x \to 2^+} g(x)$ (vi) $\lim_{x \to 2} g(x)$

(b) Sketch the graph of g.

Solution:

(a) (i) $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x = 1$

(ii) $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (2 - x^2) = 2 - 1^2 = 1$. Since $\lim_{x \to 1^-} g(x) = 1$ and $\lim_{x \to 1^+} g(x) = 1$, we have $\lim_{x \to 1} g(x) = 1$.

Note that the fact $g(1) = 3$ does not affect the value of the limit.

(iii) When $x = 1$, $g(x) = 3$, so $g(1) = 3$.

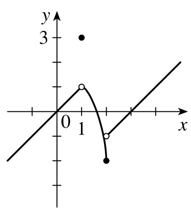
(iv) $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (2 - x^2) = 2 - 2^2 = 2 - 4 = -2$

(v) $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (x - 3) = 2 - 3 = -1$

(vi) $\lim_{x \to 2} g(x)$ does not exist since $\lim_{x \to 2^-} g(x) \ne \lim_{x \to 2^+} g(x)$.

(b)

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \le 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$



57. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

Solution:

The graph of $f(x) = [\![x]\!] + [\![-x]\!]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a. Thus, $\lim_{x \to 2^-} f(x) = -1$ and $\lim_{x \to 2^+} f(x) = -1$, so $\lim_{x \to 2} f(x) = -1$. However,

$f(2) = [\![2]\!] + [\![-2]\!] = 2 + (-2) = 0$, so $\lim_{x \to 2} f(x) \ne f(2)$.