Section 15.9 Change of Variables in Multiple Integrals

20. Use the given transformation to evaluate the integral. $\iint_R (x^2 - xy + y^2)dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$; $x = \sqrt{2}u - \sqrt{2/3}v$, $y = \sqrt{2}u + \sqrt{2/3}v$

Solution:

$x = \sqrt{2}\,u - \sqrt{2/3}\,v$, $y = \sqrt{2}\,u + \sqrt{2/3}\,v \quad \Rightarrow \quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \dfrac{4}{\sqrt{3}}$. The integrand

$x^2 - xy + y^2 = 2u^2 + 2v^2$. The planar ellipse $x^2 - xy + y^2 \le 2$ is the image of the disk $u^2 + v^2 \le 1$. Thus,

$$\iint_R (x^2 - xy + y^2)\,dA = \iint_{u^2+v^2\le 1} (2u^2 + 2v^2)\left(\tfrac{4}{\sqrt{3}}\,du\,dv\right) = \int_0^{2\pi}\int_0^1 \tfrac{8}{\sqrt{3}} r^3\,dr\,d\theta = \tfrac{4\pi}{\sqrt{3}}$$

24. An important problem in thermodynamics is to find the work done by an ideal Carnot engine. A cycle consists of alternating expansion and compression of gas in a piston. The work done by the engine is equal to the area of the region R enclosed by two isothermal curves $xy = a$, $xy = b$ and two adiabatic curves $xy^{1.4} = c$, $xy^{1.4} = d$, where $0 < a < b$ and $0 < c < d$. Compute the work done by determining the area of R.

Solution:

R is the region enclosed by the curves $xy = a$, $xy = b$, $xy^{1.4} = c$, and $xy^{1.4} = d$, so if we let $u = xy$ and $v = xy^{1.4}$ then R is the image of the rectangle enclosed by the lines $u = a$, $u = b$ $(a < b)$ and $v = c$, $v = d$ $(c < d)$. Now

$x = u/y \quad \Rightarrow \quad v = (u/y)y^{1.4} = uy^{0.4} \quad \Rightarrow \quad y^{0.4} = u^{-1}v \quad \Rightarrow \quad y = (u^{-1}v)^{1/0.4} = u^{-2.5}v^{2.5}$ and

$x = uy^{-1} = u(u^{-2.5}v^{2.5})^{-1} = u^{3.5}v^{-2.5}$, so

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3.5u^{2.5}v^{-2.5} & -2.5u^{3.5}v^{-3.5} \\ -2.5u^{-3.5}v^{2.5} & 2.5u^{-2.5}v^{1.5} \end{vmatrix} = 8.75v^{-1} - 6.25v^{-1} = 2.5v^{-1}$$

Thus the area of R, and the work done by the engine, is

$$\iint_R dA = \int_a^b\int_c^d \left|2.5v^{-1}\right|\,dv\,du = 2.5\int_a^b du\int_c^d (1/v)\,dv = 2.5\big[u\big]_a^b\,\big[\ln|v|\,\big]_c^d = 2.5(b-a)(\ln d - \ln c) = 2.5(b-a)\ln\frac{d}{c}.$$

26. Evaluate the integral by making an appropriate change of variables. $\iint_R (x+y)e^{x^2-y^2}dA$, where R is the rectangle enclosed by the lines $x - y = 0$, $x - y = 2$, $x + y = 0$, and $x + y = 3$.

Solution:

Letting $u = x + y$ and $v = x - y$, we have $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\dfrac{1}{2}$.

R is the image of the rectangle enclosed by the lines $u = 0$, $u = 3$, $v = 0$, and $v = 2$. Thus,

$$\iint_R (x+y)\,e^{x^2-y^2}\,dA = \int_0^3\int_0^2 ue^{uv}\left|-\tfrac{1}{2}\right|dv\,du = \tfrac{1}{2}\int_0^3 \left[e^{uv}\right]_{v=0}^{v=2} du = \tfrac{1}{2}\int_0^3 (e^{2u} - 1)\,du$$
$$= \tfrac{1}{2}\left[\tfrac{1}{2}e^{2u} - u\right]_0^3 = \tfrac{1}{2}\left(\tfrac{1}{2}e^6 - 3 - \tfrac{1}{2}\right) = \tfrac{1}{4}(e^6 - 7)$$

28. Evaluate the integral by making an appropriate change of variables. $\iint_R \sin(9x^2+4y^2)dA$, where R is the region in the first quadrant bounded by the ellipse $9x^2+4y^2=1$.

Solution:

Letting $u=3x$ and $v=2y$, we have $9x^2+4y^2=u^2+v^2$, $x=\frac{1}{3}u$, and $y=\frac{1}{2}v$. Then $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1/3 & 0\\ 0 & 1/2\end{vmatrix}=\dfrac{1}{6}$.

R is the image of the quarter-disk D given by $u^2+v^2\le 1$, $u\ge 0$, $v\ge 0$. Thus,

$$\iint_R \sin(9x^2+4y^2)\,dA=\iint_D \tfrac{1}{6}\sin(u^2+v^2)\,du\,dv=\int_0^{\pi/2}\int_0^1 \tfrac{1}{6}\sin(r^2)\,r\,dr\,d\theta=\tfrac{\pi}{12}\left[-\tfrac{1}{2}\cos r^2\right]_0^1=\tfrac{\pi}{24}(1-\cos 1)$$

31. Let f be continuous on $[0,1]$ and let R be the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Show that $\iint_R f(x+y)dA=\int_0^1 uf(u)du$.

Solution:

Let $u=x+y$ and $v=y$, then $x=u-v$, $y=v$, $\dfrac{\partial(x,y)}{\partial(u,v)}=1$ and R is the image under T of the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Thus

$$\iint_R f(x+y)\,dA=\int_0^1\int_0^u (1)\,f(u)\,dv\,du=\int_0^1 f(u)\big[\,v\,\big]_{v=0}^{v=u}\,du=\int_0^1 uf(u)\,du \quad \text{as desired.}$$