Section 15.8 Triple Integrals in Spherical Coordinates

10. Write the equation in spherical coordinates. (a) $z = x^2 + y^2$ (b) $z = x^2 - y^2$.

Solution:

(a) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation $z = x^2 + y^2$ becomes

$\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$ or $\rho\cos\phi = \rho^2\sin^2\phi$. If $\rho \neq 0$, this becomes $\cos\phi = \rho\sin^2\phi$

or $\rho = \cos\phi\csc^2\phi$ or $\rho = \cot\phi\csc\phi$. ($\rho = 0$ corresponds to the origin which is included in the surface.)

(b) The equation $z = x^2 - y^2$ becomes $\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2$

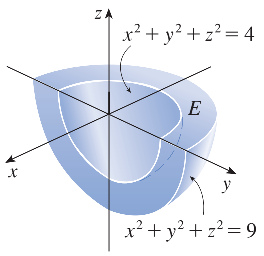
or $\rho\cos\phi = \rho^2(\sin^2\phi)(\cos^2\theta - \sin^2\theta)$ $\Leftrightarrow$ $\rho\cos\phi = \rho^2\sin^2\phi\cos 2\theta$. If $\rho \neq 0$, this becomes

$\cos\phi = \rho\sin^2\phi\cos 2\theta$. ($\rho = 0$ corresponds to the origin which is included in the surface.)

21. (a) Express the triple integral $\iiint_E f(x,y,z)\,dV$ as an iterated integral in spherical coordinates for the given function f and solid region E.

(b) Evaluate the iterated integral.

$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$



Solution:

(a) The solid can be described in spherical coordinates by $E = \{(\rho,\theta,\phi) \mid 2 \le \rho \le 3, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}, \frac{\pi}{2} \le \phi \le \pi\}$.

Thus, $\iiint_E \sqrt{x^2+y^2+z^2}\,dV = \int_{\pi/2}^{\pi}\int_{\pi/2}^{3\pi/2}\int_2^3 \rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$.

(b) $\int_{\pi/2}^{\pi}\int_{\pi/2}^{3\pi/2}\int_2^3 \rho^3\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/2}^{\pi}\sin\phi\,d\phi\int_{\pi/2}^{3\pi/2}d\theta\int_2^3\rho^3\,d\rho$

$$= \Big[-\cos\phi\Big]_{\phi=\pi/2}^{\phi=\pi}\Big[\theta\Big]_{\theta=\pi/2}^{\theta=3\pi/2}\left[\frac{\rho^4}{4}\right]_{\rho=2}^{\rho=3} = (1)(\pi)\cdot\frac{1}{4}(81-16) = \frac{65\pi}{4}$$

28. Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$, where E lies above the cone $z = \sqrt{x^2+y^2}$ and between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$.

Solution:

In spherical coordinates, the cone $z = \sqrt{x^2+y^2}$ is equivalent to $\phi = \pi/4$ (as in Example 4) and E is represented by $\{(\rho,\theta,\phi) \mid 1 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4\}$. Also, $\sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$, so

$$\iiint_E \sqrt{x^2+y^2+z^2}\,dV = \int_0^{\pi/4}\int_0^{2\pi}\int_1^2 \rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi/4}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_1^2\rho^3\,d\rho$$

$$= [-\cos\phi]_0^{\pi/4}\,\left[\theta\right]_0^{2\pi}\,\left[\tfrac{1}{4}\rho^4\right]_1^2 = \left(-\tfrac{\sqrt{2}}{2}+1\right)(2\pi)\cdot\tfrac{1}{4}(16-1) = \tfrac{15}{2}\pi\left(1-\tfrac{\sqrt{2}}{2}\right)$$

30. Use spherical coordinates. Find the average distance from a point in a ball of radius a to its center.

Solution:

If we center the ball at the origin, then the ball is given by $B = \{(\rho, \theta, \phi) \mid 0 \le \rho \le a, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$ and the distance from any point (x, y, z) in the ball to the center $(0, 0, 0)$ is $\sqrt{x^2 + y^2 + z^2} = \rho$. Thus the average distance is

$$\frac{1}{V(B)} \iiint_B \rho \, dV = \frac{1}{\frac{4}{3}\pi a^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{a} \rho \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \frac{3}{4\pi a^3} \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^{a} \rho^3 \, d\rho$$

$$= \frac{3}{4\pi a^3} \left[-\cos\phi\right]_0^{\pi} \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{4}\rho^4\right]_0^{a} = \frac{3}{4\pi a^3} (2)(2\pi)\left(\tfrac{1}{4}a^4\right) = \tfrac{3}{4}a$$