Section 15.4 Applications of Double Integrals

8. Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ. D is the triangular region enclosed by the lines $y = 0$, $y = 2x$, and $x + 2y = 1$; $\rho(x, y) = x$

Solution:

Here $D = \left\{(x, y) \mid 0 \le y \le \frac{2}{5},\ \ y/2 \le x \le 1 - 2y\right\}$.

$$m = \int_0^{2/5} \int_{y/2}^{1-2y} x\,dx\,dy = \int_0^{2/5} \left[\tfrac{1}{2}x^2\right]_{x=y/2}^{x=1-2y} dy = \tfrac{1}{2}\int_0^{2/5} \left[(1-2y)^2 - \left(\tfrac{1}{2}y\right)^2\right] dy$$
$$= \tfrac{1}{2}\int_0^{2/5} \left(\tfrac{15}{4}y^2 - 4y + 1\right) dy = \tfrac{1}{2}\left[\tfrac{5}{4}y^3 - 2y^2 + y\right]_0^{2/5} = \tfrac{1}{2}\left[\tfrac{2}{25} - \tfrac{8}{25} + \tfrac{2}{5}\right] = \tfrac{2}{25},$$

$$M_y = \int_0^{2/5} \int_{y/2}^{1-2y} x \cdot x\,dx\,dy = \int_0^{2/5} \left[\tfrac{1}{3}x^3\right]_{x=y/2}^{x=1-2y} dy = \tfrac{1}{3}\int_0^{2/5} \left[(1-2y)^3 - \left(\tfrac{1}{2}y\right)^3\right] dy$$
$$= \tfrac{1}{3}\int_0^{2/5} \left(-\tfrac{65}{8}y^3 + 12y^2 - 6y + 1\right) dy = \tfrac{1}{3}\left[-\tfrac{65}{32}y^4 + 4y^3 - 3y^2 + y\right]_0^{2/5} = \tfrac{1}{3}\left[-\tfrac{13}{250} + \tfrac{32}{125} - \tfrac{12}{25} + \tfrac{2}{5}\right] = \tfrac{31}{750},$$

$$M_x = \int_0^{2/5} \int_{y/2}^{1-2y} y \cdot x\,dx\,dy = \int_0^{2/5} y\left[\tfrac{1}{2}x^2\right]_{x=y/2}^{x=1-2y} dy = \tfrac{1}{2}\int_0^{2/5} y\left(\tfrac{15}{4}y^2 - 4y + 1\right) dy$$
$$= \tfrac{1}{2}\int_0^{2/5} \left(\tfrac{15}{4}y^3 - 4y^2 + y\right) dy = \tfrac{1}{2}\left[\tfrac{15}{16}y^4 - \tfrac{4}{3}y^3 + \tfrac{1}{2}y^2\right]_0^{2/5} = \tfrac{1}{2}\left[\tfrac{3}{125} - \tfrac{32}{375} + \tfrac{2}{25}\right] = \tfrac{7}{750}.$$

Hence $m = \frac{2}{25}$, $(\overline{x}, \overline{y}) = \left(\frac{31/750}{2/25}, \frac{7/750}{2/25}\right) = \left(\frac{31}{60}, \frac{7}{60}\right)$.

13. A lamina occupies the part of the disk $x^2 + y^2 \le 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

Solution:

$$\rho(x, y) = ky, \qquad m = \iint_D ky\,dA = \int_0^{\pi/2}\int_0^1 k(r\sin\theta)\,r\,dr\,d\theta = k\int_0^{\pi/2} \sin\theta\,d\theta \int_0^1 r^2\,dr$$
$$= k\left[-\cos\theta\right]_0^{\pi/2} \left[\tfrac{1}{3}r^3\right]_0^1 = k(1)\left(\tfrac{1}{3}\right) = \tfrac{1}{3}k,$$

$$M_y = \iint_D x \cdot ky\,dA = \int_0^{\pi/2}\int_0^1 k(r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = k\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \int_0^1 r^3\,dr$$
$$= k\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} \left[\tfrac{1}{4}r^4\right]_0^1 = k\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right) = \tfrac{1}{8}k,$$

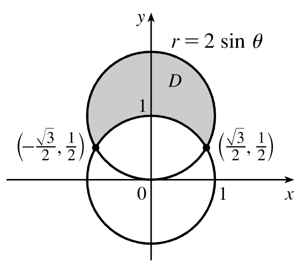
$$M_x = \iint_D y \cdot ky\,dA = \int_0^{\pi/2}\int_0^1 k(r\sin\theta)^2\,r\,dr\,d\theta = k\int_0^{\pi/2} \sin^2\theta\,d\theta \int_0^1 r^3\,dr$$
$$= k\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\tfrac{1}{4}r^4\right]_0^1 = k\left(\tfrac{\pi}{4}\right)\left(\tfrac{1}{4}\right) = \tfrac{\pi}{16}k.$$

Hence $(\overline{x}, \overline{y}) = \left(\frac{k/8}{k/3}, \frac{k\pi/16}{k/3}\right) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

18. A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

Solution:

$\rho(x, y) = k/\sqrt{x^2 + y^2} = k/r$.



$$m = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} \frac{k}{r}\,r\,dr\,d\theta = k\int_{\pi/6}^{5\pi/6} [(2\sin\theta) - 1]\,d\theta$$
$$= k\big[-2\cos\theta - \theta\big]_{\pi/6}^{5\pi/6} = 2k\left(\sqrt{3} - \tfrac{\pi}{3}\right)$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$M_x = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} kr\sin\theta\,dr\,d\theta = \tfrac{1}{2}k\int_{\pi/6}^{5\pi/6} (4\sin^3\theta - \sin\theta)\,d\theta$$
$$= \tfrac{1}{2}k\big[-3\cos\theta + \tfrac{4}{3}\cos^3\theta\big]_{\pi/6}^{5\pi/6} = \sqrt{3}\,k$$

Hence $(\overline{x}, \overline{y}) = \left(0, \frac{3\sqrt{3}}{2\left(3\sqrt{3} - \pi\right)}\right)$.