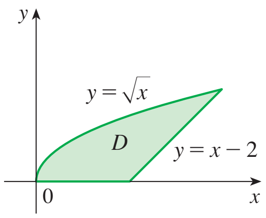
Section 15.2 Double Integrals over General Regions

9. (a) Express the double integral $\iint_D f(x, y) dA$ as an iterated integral for the given function f and region D.

   (b) Evaluate the iterated integral.



Solution:

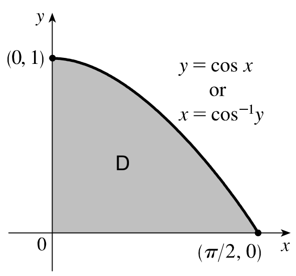
(a) We express the iterated integral as a Type II. A Type I would require the sum of two integrals. The curves intersect when $\sqrt{x} = x - 2 \quad \Rightarrow \quad x = x^2 - 4x + 4 \quad \Leftrightarrow \quad 0 = x^2 - 5x + 4 \quad \Leftrightarrow \quad (x-4)(x-1) = 0 \quad \Leftrightarrow \quad x = 1$ or $x = 4$. The point for $x = 1$ is not in D. Thus, the point of intersection of the curves is $(4, 2)$ and the integral is $\int_0^2 \int_{y^2}^{y+2} xy\, dx\, dy$.

(b) $$\int_0^2 \int_{y^2}^{y+2} xy\, dx\, dy = \int_0^2 y \left[\frac{x^2}{2}\right]_{x=y^2}^{x=y+2} dy = \frac{1}{2}\int_0^2 y\left[(y+2)^2 - (y^2)^2\right] dy = \frac{1}{2}\int_0^2 [y^3 + 4y^2 + 4y - y^5]\, dy$$
$$= \tfrac{1}{2}\left[\tfrac{1}{4}y^4 + \tfrac{4}{3}y^3 + 2y^2 - \tfrac{1}{6}y^6\right]_0^2 = \tfrac{1}{2}\left(4 + \tfrac{32}{3} + 8 - \tfrac{32}{3}\right) = 6$$

21. Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

$$\iint_D \sin^2 x dA, \;\; D \text{ is bounded by } y = \cos x, \;\; 0 \le x \le \frac{\pi}{2}, \;\; y = 0, \;\; x = 0$$

Solution:



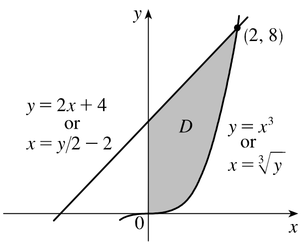
If we describe D as a type I region, $D = \{(x, y) \mid 0 \le x \le \pi/2, 0 \le y \le \cos x\}$ and $\iint_D \sin^2 x\, dA = \int_0^{\pi/2} \int_0^{\cos x} \sin^2 x\, dy\, dx$. As a type II region, $D = \{(x, y) \mid 0 \le x \le \cos^{-1} y, 0 \le y \le 1\}$ and $\iint_D \sin^2 x\, dA = \int_0^1 \int_0^{\cos^{-1} y} \sin^2 x\, dx\, dy$. Evaluating $\int_0^{\cos^{-1} y} \sin^2 x\, dx$ will result in a very difficult integral. Therefore, we evaluate the iterated integral that describes D as a type I region because integrating $\sin^2 x$ with respect to y is easy.

$$\int_0^{\pi/2} \int_0^{\cos x} \sin^2 x\, dy\, dx = \int_0^{\pi/2} \sin^2 x \left[y\right]_{y=0}^{y=\cos x} dx = \int_0^{\pi/2} \cos x \sin^2 x\, dx$$
$$= \int_0^1 u^2\, du \quad \begin{bmatrix} u = \sin x, \\ du = \cos x\, dx \end{bmatrix} \quad = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$

22. Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

$$\iint_D 6x^2\, dA, \;\; D \text{ is bounded by } y = x^3, y = 2x + 4, x = 0$$

Solution:



By inspection, the curves $y = 2x + 4$ and $y = x^3$ intersect when $x^3 = 2x + 4 \iff x = 2$, so the point of intersection is $(2, 8)$. If we describe D as a type 1 region, $D = \{(x, y) \mid 0 \le x \le 2,\, x^3 \le y \le 2x + 4\}$ and the integral is

$$\iint_D 6x^2\, dA = \int_0^2 \int_{x^3}^{2x+4} 6x^2\, dy\, dx.$$

If we describe D as a type II region, the right boundary curve is $x = \sqrt[3]{y}$, but the left boundary curve consists of two parts, $x = 0$ for $0 \le y \le 4$ and $x = y/2 - 2$ for $4 \le y \le 8$.

In either case, the resulting iterated integrals are not difficult to evaluate, but the region D is more simply described as a type I region, giving one iterated integral rather than a sum of two, so we evaluate that integral:

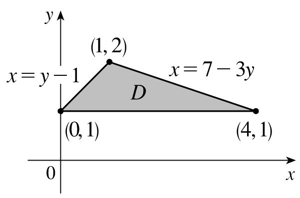
$$\int_0^2 \int_{x^3}^{2x+4} 6x^2\, dy\, dx = \int_0^2 \Big[6x^2 y\Big]_{y=x^3}^{y=2x+4} dx = \int_0^2 [6x^2(2x + 4 - x^3)]\, dx = \int_0^2 (12x^3 + 24x^2 - 6x^5)\, dx$$

$$= \big[3x^4 + 8x^3 - x^6\big]_0^2 = 48 + 64 - 64 = 48$$

25. Evaluate the double integral.

$$\iint_D y^2 dA,\;\; D \text{ is the triangular region with vertices } (0,1), (1,2), (4,1)$$

Solution:



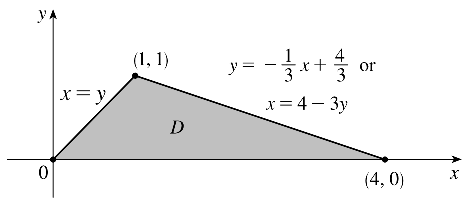
$\iint_D y^2\, dA = \int_1^2 \int_{y-1}^{7-3y} y^2\, dx\, dy = \int_1^2 \left[xy^2\right]_{x=y-1}^{x=7-3y} dy$

$= \int_1^2 [(7 - 3y) - (y - 1)]\, y^2\, dy = \int_1^2 (8y^2 - 4y^3)\, dy$

$= \left[\frac{8}{3}y^3 - y^4\right]_1^2 = \frac{64}{3} - 16 - \frac{8}{3} + 1 = \frac{11}{3}$

28. Evaluate the double integral $\iint_D y\; dA$, D is the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(4, 0)$.

Solution:



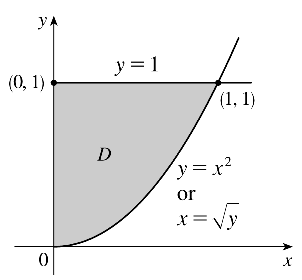
$\iint_D y\, dA = \int_0^1 \int_y^{4-3y} y\, dx\, dy$

$= \int_0^1 \left[xy\right]_{x=y}^{x=4-3y} dy = \int_0^1 (4y - 3y^2 - y^2)\, dy$

$= \int_0^1 (4y - 4y^2)\, dy = \left[2y^2 - \frac{4}{3}y^3\right]_0^1 = 2 - \frac{4}{3} - 0 = \frac{2}{3}$

62. Evaluate the integral $\int_0^1 \int_{x^2}^1 \sqrt{y} \sin y\; dydx$ by reversing the order of integration.

Solution:



$\int_0^1 \int_{x^2}^1 \sqrt{y}\, \sin y\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\, \sin y\, dx\, dy = \int_0^1 \sqrt{y}\, \sin y\, [x]_{x=0}^{x=\sqrt{y}}\, dy$

$= \int_0^1 \left(\sqrt{y}\, \sin y\right)\left(\sqrt{y} - 0\right) dy = \int_0^1 y\, \sin y\, dy$

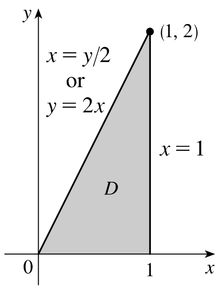
$= -y \cos y]_0^1 + \int_0^1 \cos y\, dy$

[by integrating by parts with $u = y$, $dv = \sin y\, dy$]

$= [-y \cos y + \sin y]_0^1 = -\cos 1 + \sin 1 - 0 = \sin 1 - \cos 1$

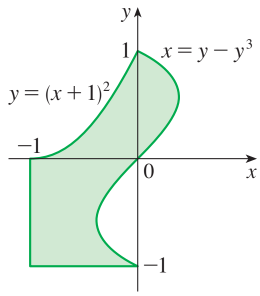
64. Evaluate the integral by reversing the order of integration. $\int_0^2 \int_{y/2}^1 y\cos(x^3-1)dxdy$

Solution:



$$\int_0^2 \int_{y/2}^1 y\cos(x^3-1)\,dx\,dy = \int_0^1 \int_0^{2x} y\cos(x^3-1)\,dy\,dx$$
$$= \int_0^1 \cos(x^3-1)\,\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=2x}\,dx$$
$$= \int_0^1 2x^2\cos(x^3-1)\,dx = \tfrac{2}{3}\sin(x^3-1)\Big]_0^1$$
$$= \tfrac{2}{3}\left[0-\sin(-1)\right] = -\tfrac{2}{3}\sin(-1) = \tfrac{2}{3}\sin 1$$

68. Express D as a union of regions of type I or type II and evaluate the integral $\iint_D y\ dA$.



Solution:

$D = \{(x,y)\mid -1\le y\le 0,\ -1\le x\le y-y^3\} \cup \{(x,y)\mid 0\le y\le 1,\ \sqrt{y}-1\le x\le y-y^3\}$, both type II.

$$\iint_D y\,dA = \int_{-1}^0\int_{-1}^{y-y^3} y\,dx\,dy + \int_0^1\int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy = \int_{-1}^0 [xy]_{x=-1}^{x=y-y^3}\,dy + \int_0^1 [xy]_{x=\sqrt{y}-1}^{x=y-y^3}\,dy$$
$$= \textstyle\int_{-1}^0 (y^2-y^4+y)\,dy + \int_0^1 (y^2-y^4-y^{3/2}+y)\,dy$$
$$= \left[\tfrac{1}{3}y^3-\tfrac{1}{5}y^5+\tfrac{1}{2}y^2\right]_{-1}^0 + \left[\tfrac{1}{3}y^3-\tfrac{1}{5}y^5-\tfrac{2}{5}y^{5/2}+\tfrac{1}{2}y^2\right]_0^1$$
$$= (0-\tfrac{11}{30}) + (\tfrac{7}{30}-0) = -\tfrac{2}{15}$$