Section 10.4 Calculus in Polar Coordinates

10. Sketch the curve and find the area that it encloses. $r = 2 + 2\cos\theta$.

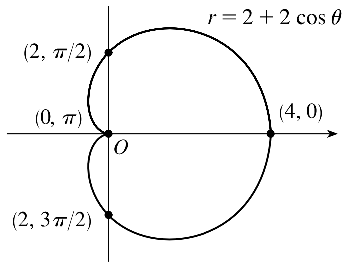
Solution:

$$A = \int_0^{2\pi} \tfrac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \tfrac{1}{2}(2 + 2\cos\theta)^2 d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}(4 + 8\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}\left[4 + 8\cos\theta + 4 \cdot \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta = \left[3\theta + 4\sin\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = 6\pi$$



28. Find the area of the region that lies inside the first curve and outside the second curve. $r = 3\sin\theta$, $r = 2 - \sin\theta$.

Solution:

$3\sin\theta = 2 - \sin\theta \quad \Rightarrow \quad 4\sin\theta = 2 \quad \Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$.

$$A = 2\int_{\pi/6}^{\pi/2} \tfrac{1}{2}\left[(3\sin\theta)^2 - (2 - \sin\theta)^2\right] d\theta$$

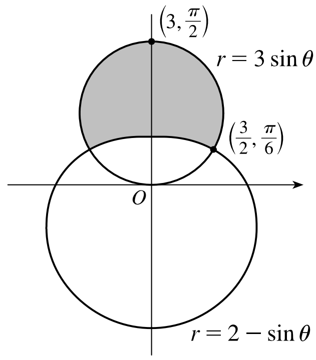
$$= \int_{\pi/6}^{\pi/2} (9\sin^2\theta - 4 + 4\sin\theta - \sin^2\theta]\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} (8\sin^2\theta + 4\sin\theta - 4)\, d\theta$$

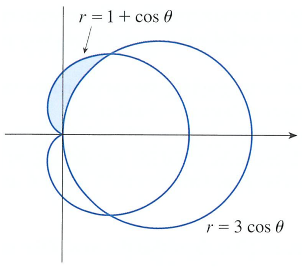
$$= 4\int_{\pi/6}^{\pi/2} \left[2 \cdot \tfrac{1}{2}(1 - \cos 2\theta) + \sin\theta - 1\right] d\theta$$

$$= 4\int_{\pi/6}^{\pi/2} (\sin\theta - \cos 2\theta)\, d\theta = 4\left[-\cos\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2}$$

$$= 4\left[(0 - 0) - \left(-\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{4}\right)\right] = 4\left(\tfrac{3\sqrt{3}}{4}\right) = 3\sqrt{3}$$



45. Find the area of the shaded region.



Solution:

$1 + \cos\theta = 3\cos\theta \quad \Rightarrow \quad 1 = 2\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$. The area swept out by $r = 1 + \cos\theta$, $\pi/3 \le \theta \le \pi$, contains the shaded region plus the portion of the circle $r = 3\cos\theta$, $\pi/3 \le \theta \le \pi/2$. Thus, the area of the shaded region is given by

$$A = \int_{\pi/3}^{\pi} \tfrac{1}{2}(1 + \cos\theta)^2\, d\theta - \int_{\pi/3}^{\pi/2} \tfrac{1}{2}(3\cos\theta)^2\, d\theta = \tfrac{1}{2}\int_{\pi/3}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta - \tfrac{9}{2}\int_{\pi/3}^{\pi/2} \cos^2\theta\, d\theta$$

$$= \tfrac{1}{2}\int_{\pi/3}^{\pi} \left[1 + 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta - \tfrac{9}{2}\int_{\pi/3}^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2}\int_{\pi/3}^{\pi} \left(\tfrac{3}{2} + 2\cos\theta + \tfrac{1}{2}\cos 2\theta\right) d\theta - \tfrac{9}{4}\int_{\pi/3}^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_{\pi/3}^{\pi} - \tfrac{9}{4}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2}$$

$$= \tfrac{1}{2}\left[\tfrac{3\pi}{2} - \left(\tfrac{\pi}{2} + \sqrt{3} + \tfrac{\sqrt{3}}{8}\right)\right] - \tfrac{9}{4}\left[\tfrac{\pi}{2} - \left(\tfrac{\pi}{3} + \tfrac{\sqrt{3}}{4}\right)\right] = \tfrac{1}{2}\left(\pi - \tfrac{9\sqrt{3}}{8}\right) - \tfrac{9}{4}\left(\tfrac{\pi}{6} - \tfrac{\sqrt{3}}{4}\right) = \tfrac{\pi}{2} - \tfrac{3\pi}{8} = \tfrac{\pi}{8}$$

68. Find the s1ope of the tangent 1ine to the given polar curve at the point specified by the value of θ.

$$r = 1 + 2\cos\theta, \quad \theta = \frac{\pi}{3}$$

Solution:

$r = 1 + 2\cos\theta \;\;\Rightarrow\;\; x = r\cos\theta = (1 + 2\cos\theta)\cos\theta,\, y = r\sin\theta = (1 + 2\cos\theta)\sin\theta \;\;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(1 + 2\cos\theta)\cos\theta + \sin\theta\,(-2\sin\theta)}{(1 + 2\cos\theta)(-\sin\theta) + \cos\theta\,(-2\sin\theta)}$$

When $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{2\left(\frac{1}{2}\right) + \left(\sqrt{3}/2\right)\left(-\sqrt{3}\right)}{2\left(-\sqrt{3}/2\right) + \frac{1}{2}\left(-\sqrt{3}\right)} \cdot \frac{2}{2} = \frac{2 - 3}{-2\sqrt{3} - \sqrt{3}} = \frac{-1}{-3\sqrt{3}} = \frac{\sqrt{3}}{9}.$