There are **7** questions in this exam.

1. (12%) Find the derivative of the given function.

   (a) (6%) $f(x) = \tan\left(x^2 e^{5x}\right)$

Solution:
We apply the chain rule and the product rule:

$$\begin{aligned}\frac{d}{dx}(f(x)) &= \frac{1}{\cos^2(x^2e^{5x})}\frac{d}{dx}(x^2e^{5x}) \\ &= \frac{(2xe^{5x} + 5x^2e^{5x})}{\cos^2(x^2e^{5x})}.\end{aligned}$$

Grading: 6 points in total. Remove the following amount of points for the following kinds of mistakes:

- Deriving a function (like tangent or exponential) wrong: remove 3 points.
- Not applying the chain rule or product rule at all: remove 6 points.
- Not applying the chain rule or product rule correctly: remove 3 points.
- Other minor computational mistakes: remove 1 point.

   (b) (6%) $g(x) = \ln\left|\sin\left(e^{(x^2)}\right)\right|$

Solution:
We apply the chain rule:

$$\begin{aligned}\frac{d}{dx}(g(x)) &= \frac{1}{\sin\left(e^{(x^2)}\right)}\frac{d}{dx}\left(\sin\left(e^{(x^2)}\right)\right) \\ &= \frac{\cos\left(e^{(x^2)}\right)}{\sin\left(e^{(x^2)}\right)}\frac{d}{dx}\left(e^{(x^2)}\right) \\ &= \frac{\cos\left(e^{(x^2)}\right)}{\sin\left(e^{(x^2)}\right)}2xe^{x^2}.\end{aligned}$$

Grading: 6 points in total. Remove the following amount of points for the following kinds of mistakes:

- Deriving a function (like natural logarithm or sine) wrong: remove 3 points.
- Not applying the chain rule at all: remove 6 points.
- Not applying the chain rule correctly: remove 3 points.
- Other minor computational mistakes: remove 1 point.

2. (24%) Evaluate the following limits.

   (a) (6%) $\lim_{x\to\infty} e^{\sqrt{x}-\sqrt{x-1}}$

Solution:
We know that

$$\lim_{x\to\infty} \mathrm{e}^{\sqrt{x}-\sqrt{x-1}} = \mathrm{e}^{\lim_{x\to\infty}\sqrt{x}-\sqrt{x-1}}$$

thus it suffices to know the limit $\lim_{x\to\infty}\sqrt{x}-\sqrt{x-1}$.

Method 1: rationalization

$$\lim_{x\to\infty} \sqrt{x}-\sqrt{x-1} = \lim_{x\to\infty} (\sqrt{x}-\sqrt{x-1}) \cdot \left(\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}\right) \text{ (2 pts for rationalization)}$$

$$= \lim_{x\to\infty} \frac{x-(x-1)}{\sqrt{x}+\sqrt{x-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}+\sqrt{x-1}} = 0. \text{ (2 pts for correct limit calculation)}$$

Method 2: l'Hospital's rule

$$\lim_{x\to\infty} \sqrt{x}-\sqrt{x-1} = \lim_{x\to\infty} \sqrt{x}\Big(1-\frac{\sqrt{x-1}}{\sqrt{x}}\Big) \text{ (Type } 0\cdot\infty) \text{ (1 pt for factor out)}$$

$$= \lim_{x\to\infty} \frac{(1-\frac{\sqrt{x-1}}{\sqrt{x}})}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{(1-\sqrt{1-\frac{1}{x}})}{x^{-\frac{1}{2}}} \text{ (Type } \frac{0}{0}) \text{ (1 pt for changing it to the correct type)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2}(1-\frac{1}{x})^{-\frac{1}{2}}(-x^{-2})}{-\frac{1}{2}x^{-\frac{3}{2}}} \text{ (l'Hospital's rule) (1 pt for correctly use the l'Hospital's rule)}$$

$$= \lim_{x\to\infty} \frac{1}{x^{\frac{1}{2}}(1-\frac{1}{x})^{\frac{1}{2}}} = 0. \text{ (1 pt for the correct limit)}$$

Thus,

$$\lim_{x\to\infty} \mathrm{e}^{\sqrt{x}-\sqrt{x-1}} = \mathrm{e}^{\lim_{x\to\infty}\sqrt{x}-\sqrt{x-1}} = \mathrm{e}^0 = 1. \text{ (2 pts for obtaining the limit)}$$

Remark 0.1. There are also other approaches to method 2 in Problem 2(a). For example, we can also factor out $\sqrt{x-1}$, or change it to type $\frac{\infty}{\infty}$, e.g.

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\frac{1}{1-\frac{\sqrt{x-1}}{\sqrt{x}}}}.$$

(b) (6%) $\lim_{x\to 0} e^{3x}\cdot|\sin(x)|\cdot\cos\left(\frac{\pi}{x}\right)$

Solution:
Let $g(x) = \mathrm{e}^{3x}|\sin(x)|\cos(\frac{\pi}{x})$. We know that $-1 \le \cos(\frac{\pi}{x}) \le 1$ for all $x \ne 0$. Define the following functions

$$h(x) = \mathrm{e}^{3x}|\sin(x)|, \ f(x) = -\mathrm{e}^{3x}|\sin(x)|. \text{ (1 pt for defining the functions)}$$

Then, since $\mathrm{e}^{3x}|\sin(x)| \ge 0$ for all real number x, we have

$$f(x) = -\mathrm{e}^{3x}|\sin(x)| \le g(x) = \mathrm{e}^{3x}|\sin(x)|\cos(x) \le h(x) = \mathrm{e}^{3x}|\sin(x)| \text{ (1 pt for this inequality)}$$

for all real number x.
By the limit laws and continuity, we know that

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \mathrm{e}^{3x}|\sin(x)| = (\lim_{x\to 0}\mathrm{e}^{3x})\cdot(\lim_{x\to 0}|\sin(x)|) = 1\cdot 0 = 0$$

and similarly $\lim_{x\to 0} f(x) = \lim_{x\to 0}(-h(x)) = 0$ (1 pt for each the limit calculation of $h(x)$ and $f(x)$).
By the squeeze theorem, we conclude that

$$\lim_{x\to 0} g(x) = 0. \text{ (1 pt for stating the squeeze theorem, 1 pt for the correct limit)}$$

(c) (6%) $\lim_{x\to 0} \frac{1-\cos(3x)}{\sqrt{x^2+4}-2}$

Solution:

$$\lim_{x\to 0} \frac{1-\cos(3x)}{\sqrt{x^2+4}-2} = \lim_{x\to 0} \frac{1-\cos(3x)}{\sqrt{x^2+4}-2}\cdot\frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} \text{ (1 pt for the first rationalization)}$$

$$= \lim_{x\to 0} \frac{(1-\cos(3x))(\sqrt{x^2+4}+2)}{(x^2+4)-4} = \lim_{x\to 0} \frac{(1-\cos(3x))(\sqrt{x^2+4}+2)}{x^2}$$

$$= \lim_{x\to 0} \frac{(1-\cos(3x))(\sqrt{x^2+4}+2)}{x^2} \cdot \frac{1+\cos(3x)}{1+\cos(3x)} \text{ (1 pt for the second rationalization)}$$

$$= \lim_{x\to 0} \frac{\sqrt{x^2+4}+2}{x^2} \cdot \frac{1-\cos^2(3x)}{1+\cos(3x)}$$

$$= \lim_{x\to 0} \frac{\sin^2(3x)}{x^2} \cdot \frac{\sqrt{x^2+4}+2}{1+\cos(3x)} \text{ (1 pt for simplification to this form)}$$

Since

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} 3\left(\frac{\sin(3x)}{3x}\right) = \lim_{\theta\to 0} 3\frac{\sin\theta}{\theta} = 3 \ (\theta = 3x) \text{ (1 pt for the limit calculation)}$$

and

$$\lim_{x\to 0} \frac{\sqrt{x^2+4}+2}{1+\cos(3x)} = \frac{\sqrt{4}+2}{1+1} = \frac{4}{2} = 2. \text{ (1 pt for the limit calculation)}$$

We conclude that

$$\lim_{x\to 0} \frac{\sin^2(3x)}{x^2} \cdot \frac{\sqrt{x^2+4}+2}{1+\cos(3x)} = 3\cdot 3\cdot 2 = 18. \text{ (1 pt for the correct answer)}$$

(d) (6%) $\lim_{x\to -\infty} x\ln\left(1-\frac{4}{x}\right)$

Solution:

$$\lim_{x\to -\infty} x\ln(1-\frac{4}{x}) \text{ (type } 0\cdot\infty)$$

$$= \lim_{x\to -\infty} \frac{\ln(1-\frac{4}{x})}{\frac{1}{x}} \text{ (type } \frac{0}{0}) \text{ (2 pts for the correct indeterminate form)}$$

$$= \lim_{x\to -\infty} \frac{\frac{1}{1-\frac{4}{x}}\frac{4}{x^2}}{\frac{-1}{x^2}} \text{ (l'Hospital's rule) (2 pts for using the l'Hospital's rule and correct differentiation)}$$

$$= \lim_{x\to -\infty} \frac{-4}{1-\frac{4}{x}} = -4 \text{ (2 pts for the correct limit)}$$

3. (15%) Consider the curve defined by $y^3 + 5yx^2 + 8x^3 = 14$.

(a) (7%) Find y'.

(b) (4%) Find an equation of the tangent line through the point $(1,1)$.

(c) (4%) Consider the implicit function $y = f(x)$ given by the curve near $(1,1)$. Use linear approximation to estimate $f(0.996)$.

Solution:

(a)

$$y^3 + 5yx^2 + 8x^3 = 14$$
$$3y^2y' + 5y'x^2 + 10yx + 24x^2 = 0$$
$$(3y^2+5x^2)y' = -10yx - 24x^2$$
$$y' = -\frac{2x(5y+12x)}{3y^2+5x^2}$$

(b)

$$y'\,|_{(1,1)} = -\frac{17}{4}$$
$$y - 1 = -\frac{17}{4}(x-1)$$
$$4y + 17x = 21$$

$$y = -\frac{17}{4}x + \frac{21}{4}$$

(c)

$$f(0.996) \approx 1 - \frac{17}{4}(0.996 - 1) = 1 + 0.017 = 1.017$$

Grading:

(a) Correct idea 1 pt. Each mistake is -2 pts.

(b) Uses answer of (a). Each mistake is -2 pts.

(c) Uses answer of (b). Each mistake is -2 pts.

4. (9%) Let $f(x) = \dfrac{\sqrt{x+4}}{x}$. We know $f(5) = 0.6$.

   (a) (5%) Use linear approximation to estimate $f(5.003)$.

   (b) (4%) Is your estimate in part (a) too large or too small? Explain.

Solution:

(a)

$$f(x) = \frac{\sqrt{x+4}}{x}$$

$$f'(x) = \frac{\frac{1}{2}(x+4)^{-1/2} \cdot x - (x+4)^{1/2}}{x^2} = \frac{x - 2x - 8}{2x^2\sqrt{x+4}}$$

$$f'(5) = -\frac{13}{150}$$

$$f(5.003) \approx f(5) + f'(5)(5.003 - 5) = 0.6 - \frac{13}{50} \cdot 0.001 = 0.6 - 0.00026 = 0.59974$$

(b)

$$f(x) = \frac{\sqrt{x+4}}{x} = x^{-1}(x+4)^{1/2}$$

$$f'(x) = -x^{-2}(x+4)^{1/2} + \frac{1}{2}x^{-1}(x+4)^{-1/2}$$

$$f''(x) = 2x^{-3}(x+4)^{1/2} - x^{-2}(x+4)^{-1/2} - \frac{1}{4}x^{-1}(x+4)^{-3/2} = \frac{8(x+4)^2 - 4x(x+4) - x^2}{4x^3(x+4)^{3/2}}$$

$$= \frac{3x^2 + 48x + 128}{4x^3(x+4)^{3/2}} = \frac{3(x+8)^2 - 64}{4x^3(x+4)^{3/2}}$$

$$f''(x) > 0 \text{ for } x > 0$$

The linear approximation is an underestimate. It is too small.

Grading:

(a) 2 pts for derivative. 2 pts for tangent line or linearization. 1 pt for estimate.

(b) There can be other methods. Read the students work and -2 pts for each mistake.

5. (12%) The demand for Taipei Dome tickets to watch the Taiwan Series is given by the function $p(y) = \left(100 - \dfrac{y}{10}\right)^2$, where p is the price and y is the quantity demanded.

   (a) (4%) Find the interval of y on which $p(y)$ is decreasing, and both p and y are positive.

   (b) (4%) Find the point elasticity when $p = 100$. The formula for elasticity is $\varepsilon = \dfrac{p}{y} \cdot \dfrac{dy}{dp}$ or $\dfrac{p}{y} \cdot \dfrac{1}{dp/dy}$.

   (c) (4%) The current price is $p = 100$. Does increasing the price make the revenue $p \cdot y$ larger?

Solution:

(a) Note that $p(y) = 0 \iff y = 1000$. And $y, p > 0$. The graph of $p(y)$ is a parabola. And $p'(y) = \frac{-1}{5}(100 - \frac{y}{10}) = \frac{y}{50} - 20$. We have

$$p'(y) < 0 \iff y < 1000.$$

Thus, p is decreasing on $(0, 1000)$. (4p)

(1p for the evaluate $p'(y)$. 2p for the applying first derivative test. 1p for the answer.)

Remark: The student who use the well-known parabola property to answer the question can get 4p.

Remark: Every open intervals and closed intervals I such that $I \subset (0, 1000)$ are correct answer.

(b)

Method 1.

Let $0 < y < 1000$. The function $Y(p)$ satisfies

$$p = (100 - \frac{Y(p)}{10})^2$$

which is equivalent to $Y(p) = 10(100 - \sqrt{p})$. The derivative is $Y'(p) = \frac{-5}{\sqrt{p}}$. The point elasticity is

$$\varepsilon(p) = \frac{p \cdot Y'(p)}{Y(p)} = \frac{-5\sqrt{p}}{10(100 - \sqrt{p})}.$$

Thus, $\varepsilon(100) = \frac{-50}{900} = \frac{-1}{18}$.

(1p for the $Y(p)$. 1p for the $Y'(p)$. 1p for the $\varepsilon(p)$. 1p for the $\varepsilon(100)$.)

Method 2.

By direct calculus we have

$$p'(y) = \frac{y}{50} - 20.$$

Thus,

$$\varepsilon(y) = \frac{p}{y} \frac{1}{\frac{y}{50} - 20} = \frac{(100 - \frac{y}{10})^2}{y} \frac{1}{\frac{y}{50} - 20}.$$

When $p = 100$ we have $y = 900$. Thus,

$$\varepsilon(900) = \frac{100}{900} \frac{1}{\frac{900}{50} - 20} = \frac{-1}{18}.$$

(1p for $p'(y)$. 1p for $\varepsilon(y)$. 1p for $(p, y) = (100, 900)$. 1p for answer.

Remark: If a student make a mistake in differentiation but uses the correct formula to arrive at the wrong answer, they will receive 2p.

(c)

Method 1.

$\varepsilon = \frac{-1}{18}$. The ticket is **inelastic**. The owner should increase the price to increase the revenue.

(1p for applying $\varepsilon = \frac{-1}{18}$. 3p for the conclusion.)

Method 2.

Assume $p = p(y)$. The revenue is

$$R(y) = y \cdot p(y) = y(100 - \frac{y}{10})^2, 0 \le y \le 1000.$$

Note that $R(0) = R(1000) = 0$. And we also have

$$R'(y) = (100 - \frac{y}{10})^2 - \frac{y}{5}(100 - \frac{y}{10}).$$

And $R'(y_0) = 0 \iff y_0 = \frac{1000}{3}$. In this case, $p(y_0) = (100 - \frac{100}{3})^2 = \frac{40000}{9}$. Thus, the revenue attain its maximum at $\frac{40000}{9} > 100$. The owner should increase the price to increase the revenue.

(1p for evaluate revenue function. 1p for the $R'(y)$. 1p for the Closed Interval Method. 1p for the conclusion.)

Method 3.

Assume $y = Y(p) = 10(100 - \sqrt{p})$. The revenue is

$$R(p) = p \cdot Y(p) = 10p(100 - \sqrt{p}) = 1000p - 10p^{\frac{3}{2}},\ 0 \le p \le 10000.$$

Note that $R(0) = R(10000) = 0$. We also have

$$R'(p) = 1000 - 15\sqrt{p}.$$

Thus, $R'(p_0) = 0 \iff p_0 = (\frac{200}{3})^2 = \frac{40000}{9}$. Therefore, the revenue attain its maximum at $p_0 = \frac{40000}{9} > 100$. The owner should increase the price to increase the revenue.

(1p for evaluate revenue function. 1p for the $R'(p)$. 1p for the Closed Interval Method. 1p for the conclusion.)

6. (12%) The cost function when x units are produced and corresponding price function are given.

$$C(x) = x^3 - 12x^2 + 50x + F \quad , \quad p(x) = 890 - 3x,$$

where F is a constant.

(a) (6%) If the average cost function is minimized when $x = 16$, find the value of F.

(b) (6%) Find the amount of units to produce to maximize the profit $\Pi(x) = xp(x) - C(x)$.

Solution:

(a) The average cost function is

$$AC(x) = \frac{C(x)}{x} = x^2 - 12x + 50 + \frac{F}{x},\ x > 0.$$

Note that $AC'(x) = 2x - 12 - \frac{F}{x^2} = 0$ when $x = 16$. Thus,

$$20 - \frac{F}{256} \iff F = 5120.$$

Note that $AC''(x) = 2 + \frac{2F}{x^3} > 0$ for $x > 0$. And

$$\lim_{x\to+\infty} AC(x) = \lim_{x\to 0^+} AC(x) = +\infty.$$

The average cost function is minimize at $x = 16$.

(2p for the definition of $AC(x)$. 1p for correct $AC(x)$. 1p for correct $AC'(x)$. 1p for solving $AC'(x) = 0$. 1p for F.)

Remark: If the student get wrong $AC(x)$ but provide the correct argument of the minimum value, then the student get 3p.

(b) For $x > 0$. The profit function is

$$\begin{aligned}\Pi(x) &= xp(x) - C(x) = x(890 - 3x) - (x^3 - 12x^2 + 50x + 5120)\\ &= -x^3 + 9x^2 + 840x - 5120\end{aligned}$$

By direct calculus, we have

$$\Pi'(x) = -3x^2 + 18x + 840 \text{ and } \Pi''(x) = -6x + 18.$$

Solve $\Pi'(x) = 0$ on $x > 0$. We have

$$-3x^2 + 18x + 840 = 0 \iff x_M = 20.$$

And $\Pi''(20) = -102$. Note that $\Pi(x)$ is increasing on $(0, 20)$ and $\Pi(x)$ is decreasing on $(20, +\infty)$. Thus, the profit attain its maximum at $x_M = 20$.

(2p for correct $\Pi(x)$. 1p for $\Pi'(x)$. 1p for $\Pi''(x)$. 1p for the argument of the maximum value. (or 3p for the increasing/decreasing intervals) 1p for the correct x_M.)

Remark: If the student get wrong $\Pi(x)$ but provide the correct argument of the maximum value, then the student get 3p.

7. (16%) Let $f(x) = \dfrac{1}{(1+e^x)^2}$.

(a) Find **ALL** asymptotes of $y = f(x)$.

Solution:

(i) Vertical asymptotes: Since $1 + e^x > 0$ for all $x \in \mathbb{R}$, the domain of $f(x)$ is all $\mathbb{R}$. Moreover, f is continuous in all its domain. Therefore, there are no vertical asymptotes.

(ii) Horizontal asymptotes: We have
$$\lim_{x\to\infty} f(x) = 0,$$
so $y = 0$ is a horizontal asymptote when $x \to \infty$. And
$$\lim_{x\to-\infty} f(x) = 1,$$
so $y = 1$ is a horizontal asymptote when $x \to -\infty$.

(iii) Slant asymptotes: Since $f(x)$ has horizontal asymptotes both when $x \to \infty$ and when $x \to -\infty$, there are no slant asymptotes.

Grading: 4 points in total. Remove the following amount of points for the following kinds of mistakes:

- Remove 2 points for a wrong answer in part (i).
- Remove 2 points for a wrong answer in part (ii).

(b) Find $f'(x)$ and the intervals of increase or decrease.

Solution:
First we compute
$$\begin{aligned} f'(x) &= -2(1+e^x)^{-3}e^x \\ &= \frac{-2e^x}{(1+e^x)^3}. \end{aligned}$$

We have $-2e^x < 0$ for all $x \in \mathbb{R}$ and $(1+e^x)^3 > 0$ for all $x \in \mathbb{R}$, so $f'(x) < 0$ for all $x \in \mathbb{R}$. Therefore, f is decreasing on the interval $(-\infty, \infty)$.
Grading: 4 points in total. Remove the following amount of points for the following kinds of mistakes:

- Remove 2 points for computing $f'(x)$ wrong.
- Remove 2 points for wrong conclusion about increasing/decreasing intervals.

(c) Find $f''(x)$ and the intervals of concavity.

Solution:

First we compute

$$\begin{aligned} f''(x) &= \frac{-2e^x(1+e^x)^3 - (-2e^x)3(1+e^x)^2e^x}{(1+e^x)^6} \\ &= \frac{-2e^x(1+e^x) + 6e^xe^x}{(1+e^x)^4} \\ &= \frac{e^x(6e^x - 2(1+e^x))}{(1+e^x)^4} \\ &= \frac{e^x(4e^x-2)}{(1+e^x)^4} \\ &= \frac{2e^x(2e^x-1)}{(1+e^x)^4}. \end{aligned}$$

We have $2e^x > 0$ and $(1+e^x)^4 > 0$ for all $x \in \mathbb{R}$. For $(2e^x - 1)$, we have

$$\begin{aligned} 2e^x - 1 > 0 &\Leftrightarrow 2e^x > 1 \\ &\Leftrightarrow e^x > \frac{1}{2} \\ &\Leftrightarrow x > \ln(1) - \ln(2) \\ &\Leftrightarrow x > -\ln(2). \end{aligned}$$

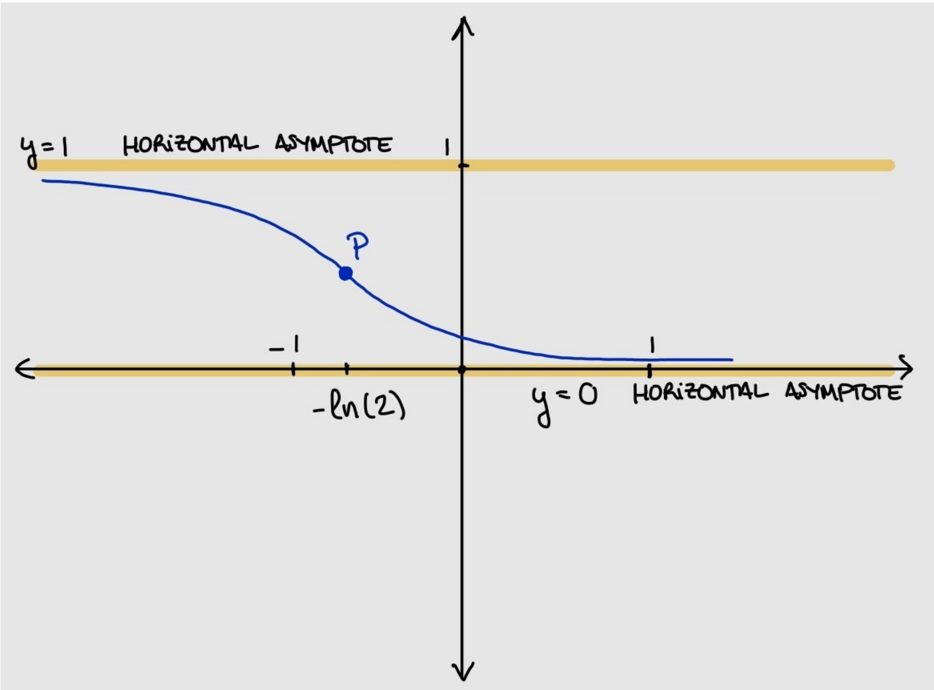
Similarly, $2e^x - 1 < 0$ if and only if $x < -\ln(2)$, and $2e^x - 1 = 0$ if and only if $x = -\ln(2)$. Therefore, the graph of f is concave upwards on the interval $(-\ln(2), \infty)$, and it is concave downwards on the interval $(-\infty, -\ln(2))$. The point $P := (-\ln(2), f(-\ln(2)))$ is an inflection point.

Grading: 4 points in total. Remove the following amount of points for the following kinds of mistakes:

- Remove 2 points for computing $f''(x)$ wrong.
- Remove 2 points for wrong conclusion about concavity intervals.

(d) Sketch the graph. Label asymptotes, any local extrema and/or inflection point(s).

Solution:
There aren't any local extrema, because f is constantly decreasing, and the only inflection point is labelled as P in the graph. The two horizontal asymptotes are drawn in yellow.



Grading: 4 points in total. Remove the following amount of points (or until you have removed 4 points) for the following kinds of mistakes:

- Remove 2 points if the graph is not always decreasing.
- Remove 2 points if the horizontal asymptotes aren't indicated or if the graph crosses them.
- Remove 2 points if the inflection point isn't marked.