112 模組05-09班 微積分4 期考解答和評分標準

1. Let S be the part of the surface $z = \tan^{-1}\left(\frac{y}{x}\right)$ in the first octant satisfying $1 \le x^2 + y^2 \le 4$ and $x \ge y \ge 0$.

   (a) (4%) Parametrize the surface S.

   (b) (10%) Evaluate the surface integral $\iint_S \sqrt{x^2+y^2}\, dS$.

Solution:

(a) Here is a list of possible parametrizations we expect students to use.

- $\mathbf{r}(x,y) = \left\langle x, y, \tan^{-1}\left(\frac{y}{x}\right)\right\rangle$, with $1 \le x^2+y^2 \le 4$ and $x \ge y \ge 0$.
- $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, with $1 \le u \le 2$ and $0 \le v \le \frac{\pi}{4}$.
- $\mathbf{r}(z,x) = \langle x, x\tan z, z\rangle$, with $0 \le z \le \frac{\pi}{4}$ and $\cos z \le x \le 2\cos z$.

(b) Here we compute the surface integral with all three parametrizations above.

$$\mathbf{r}_x = \left\langle 1, 0, \frac{-y}{x^2+y^2}\right\rangle$$

$$\mathbf{r}_y = \left\langle 0, 1, \frac{x}{x^2+y^2}\right\rangle$$

$$\mathbf{r}_x \times \mathbf{r}_y = \left\langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, 1\right\rangle$$

$$dS = \sqrt{\frac{y^2}{(x^2+y^2)^2} + \frac{x^2}{(x^2+y^2)^2} + 1} = \frac{\sqrt{1+x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$\iint_S \sqrt{x^2+y^2}\ dS = \iint_R \sqrt{1+x^2+y^2}\ dA = \int_0^{\pi/4}\int_1^2 \sqrt{1+r^2}\ r\ dr\ d\theta$$

$$= \frac{\pi}{4}\left(\frac{5\sqrt{5}-2\sqrt{2}}{3}\right) = \frac{\pi(\sqrt{125}-\sqrt{8})}{12}$$

$$\mathbf{r}_u = \langle \cos v, \sin v, 0\rangle$$

$$\mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle$$

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$$

$$dS = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$$

$$\iint_S \sqrt{x^2+y^2}\ dS = \int_0^{\pi/4}\int_1^2 u\sqrt{1+u^2}\ du\ dv$$

$$= \frac{\pi}{4}\left(\frac{5\sqrt{5}-2\sqrt{2}}{3}\right) = \frac{\pi(\sqrt{125}-\sqrt{8})}{12}$$

$$\mathbf{r}_z = \left\langle 0, x\sec^2 z, 1\right\rangle$$

$$\mathbf{r}_x = \langle 1, \tan z, 0\rangle$$

$$\mathbf{r}_z \times \mathbf{r}_x = \left\langle -\tan z, 1, -x\sec^2 z\right\rangle$$

$$dS = \sqrt{\tan^2 z + 1 + x^2\sec^4 z} = \sec z\sqrt{1+x^2\sec^2 z}$$

$$\iint_S \sqrt{x^2+y^2}\ dS = \int_0^{\pi/4}\int_{\cos z}^{2\cos z} \sqrt{x^2+x^2\tan^2 z}\ \sec z\sqrt{1+x^2\sec^2 z}\ dx\ dz$$

$$= \int_0^{\pi/4}\int_{\cos z}^{2\cos z} x\sec^2 z\sqrt{1+x^2\sec^2 z}\ dx\ dz = \int_0^{\pi/4}\left[\frac{1}{3}(1+x^2\sec^2 z)^{3/2}\right]_{\cos z}^{2\cos z}\ dz$$

$$= \frac{\pi}{4}\left(\frac{5\sqrt{5} - 2\sqrt{2}}{3}\right) = \frac{\pi(\sqrt{125} - \sqrt{8})}{12}$$

□

Grading:

- For (a), it is important to remember a parametrization is not complete without the bounds. However, note that the student's answer might be written in (b).
- (2%) for having a vector function of two variables that satisfy the equation $z = \tan^{-1}\left(\frac{y}{x}\right)$, no partial credit. But if this part is wrong, students cannot get the points for the bounds unless it is very similar to one of the given parametrizations.
- (2%) for having correct bounds that satisfy $1 \le x^2 + y^2 \le 4$ and $x \ge y \ge 0$. Each minor mistake is (-1%). As said above, if the first step of the parametrization is far from correct, then no points here.
- For (b), (5%) for finding dS, (2%) for substituting $\sqrt{x^2 + y^2}$ correctly, and (3%) for the double integral. The difficulty in grading (b) comes from students with wrong answers in (a).
- Each minor mistake in finding dS is (-1%). (-2%) for each missing step. The points (5%) and (2%) for $\sqrt{x^2 + y^2}$ can be earned even if (a) is incorrect (only if student wrote a vector function in (a)).
- If the answer in (a) is really wrong, then no points for the double integral. Otherwise the double integral follows the simple (-1%) for each minor mistake and (-2%) for each major mistake.

2. Let $\mathbf{F}(x,y) = \left(e^x - \dfrac{2y}{x^2+y^2}\right)\mathbf{i} + \left(-e^y + \dfrac{2x}{x^2+y^2}\right)\mathbf{j}$.

(a) (7%) Is $\mathbf{F}$ conservative on the upper half plane $y > 0$? If yes, find the function f on the upper half plane $y > 0$ with $f(0,1) = -e$ such that $\nabla f = \mathbf{F}$.

(b) (2%) Evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ where C_1 is the line segment from $(1,1)$ to $(2,2)$.

(c) Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is a counterclockwise simple closed curve in $\mathbb{R}\setminus\{(0,0)\}$ according to the following two cases. (i) (2%) Case 1: C does not enclose the point $(0,0)$. (ii) (8%) Case 2: C encloses the point $(0,0)$.

(d) (1%) Is $\mathbf{F}$ conservative on $\mathbb{R}\setminus\{(0,0)\}$?

Solution:

(a) Set $P(x,y) = e^x - \dfrac{2y}{x^2+y^2}$ and $Q(x,y) = -e^y + \dfrac{2x}{x^2+y^2}$. Then we compute that

$$\frac{\partial P}{\partial y} = -\frac{2(x^2+y^2) - 2y(2y)}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2},$$
$$\frac{\partial Q}{\partial x} = \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}.$$

Since P and Q have continuous first-order partial derivatives and $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on the upper half plane $y > 0$ and the upper half plane $y > 0$ is an simple-connected region, we have $\mathbf{F}$ is conservative. **(2%)**
From $\nabla f = \mathbf{F}$, we have

$$f_x = e^x - \frac{2y}{x^2+y^2}, \quad f_y = -e^y + \frac{2x}{x^2+y^2}. \ \textbf{(1\%)}$$

Integrating the first equation with respect to x, we obtain

$$f(x,y) = e^x - 2\tan^{-1}\left(\frac{x}{y}\right) + g(y). \ \textbf{(2\%)}$$

Then we differentiate $f(x,y)$ with respect to y to get

$$f_y = \frac{2x}{x^2+y^2} + g'(y).$$

So we have $g'(y) = -e^y$ and $g(y) = -e^y + K$ where K is some constants. **(1%)**
Since $f(0,1) = -e$, we have $K = -1$. Therefore,

$$f(x,y) = e^x - e^y - 2\tan^{-1}\left(\frac{x}{y}\right) - 1. \ \textbf{(1\%)}$$

(b) From (a), we have $\mathbf{F}$ is conservative and C_1 is a smooth curve on the upper half plane $y > 0$. By the Fundamental Theorem for line integrals **(1%)**, we have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(2,2) - f(1,1) = 0. \ \textbf{(1\%)}$$

(c)-(i) For any simple closed curve C in $\mathbb{R}^2 \setminus \{(0,0)\}$ that does not enclosed $(0,0)$. Let D be the region bounded by C. By Green's theorem and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ on D **(1%)**, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 0\, dA = 0. \ \textbf{(1\%)}$$

(c)-(ii) Now we consider C is a positively oriented simple closed curve in $\mathbb{R}^2 \setminus \{(0,0)\}$ that enclosed $(0,0)$. Set $C_r : x^2 + y^2 = r^2$ where r is small enough such that C_r is inside C and D is the region bounded by C and C_r. **(1%)** We parametrize C_r by $\langle r\cos\theta, r\sin\theta\rangle$, $0 \le \theta \le 2\pi$. **(1%)** Then

$$\begin{aligned}
\oint_{C_r} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(r\cos\theta, r\sin\theta) \cdot \langle -r\sin\theta, r\cos\theta\rangle\, d\theta \ \textbf{(2\%)} \\
&= \int_0^{2\pi} \langle e^{r\cos\theta} - \frac{2\sin\theta}{r}, -e^{r\sin\theta} + \frac{2\cos\theta}{r}\rangle \cdot \langle -r\sin\theta, r\cos\theta\rangle\, d\theta \\
&= \int_0^{2\pi} \left(-r\sin\theta\, e^{r\cos\theta} - r\cos\theta\, e^{r\sin\theta} + 2\right) d\theta \ \textbf{(1\%)} \\
&= e^{r\cos\theta} - e^{r\sin\theta} + 2\theta\Big]_0^{2\pi} = (e^r - 1 + 4\pi) - (e^r - 1) = 4\pi. \ \textbf{(1\%)}
\end{aligned}$$

Since P and Q have continuous first-order partial derivatives and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ on D, by Green's theorem, we obtain

$$0 = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C \mathbf{F} \cdot d\mathbf{r} - \oint_{C_r} \mathbf{F} \cdot d\mathbf{r} \ \mathbf{(1\%)}$$

Therefore, we obtain that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_r} \mathbf{F} \cdot d\mathbf{r} = 4\pi. \ \mathbf{(1\%)}$$

(d) From (c)-(ii), we have $\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi \neq 0$ **(1%)** for a closed curve in $\mathbb{R}^2 \setminus \{(0,0)\}$ that enclosed $(0,0)$. Therefore, $\mathbf{F}$ is not conservative on $\mathbb{R} \setminus \{(0,0)\}$.

3. Let $\mathbf{F}(x, y, z) = (2x - y)\,\mathbf{i} + (2y + z)\,\mathbf{j} + x^2\,\mathbf{k}$. Consider surfaces S_1 and S_2.
S_1 is the part of the ellipsoid $2x^2 + y^2 + z^2 = 4$ above the plane $z = y + 2$ with upward orientation.
S_2 is the part of the plane $z = y + 2$ inside the ellipsoid $2x^2 + y^2 + z^2 = 4$ with upward orientation.

(a) (5%) Parametrize the surface S_2.

(b) (10%) Compute $\iint_{S_2} \text{curl}\mathbf{F} \cdot d\mathbf{S}$ directly.

(c) (2%) Find $\iint_{S_1} \text{curl}\mathbf{F} \cdot d\mathbf{S}$.

Solution:

(a) Since (x, y, z) is in $2x^2 + y^2 + z^2 \leq 4$ on $z = y + 2$, we can deduce that x and y satisfy

$$\begin{aligned} & 2x^2 + y^2 + z^2 \leq 4 \\ \Leftrightarrow\ & 2x^2 + y^2 + (y+2)^2 \leq 4 \\ \Leftrightarrow\ & x^2 + (y+1)^2 \leq 1 \end{aligned}$$

The parameterized surface of S_2 is $r(x, y) = x\mathbf{i} + y\mathbf{j} + y + 2\mathbf{k}$ with $(x, y) \in D = \{(x, y) | -1 \leq x \leq -\sqrt{1 - x^2} - 1 \leq y \leq \sqrt{1 - x^2} - 1\}$.

(1pt) Find D.

(2pt) Simplify D to a circle. They need this in Question (b), the grader has to check if they put their answer in Question (b).

(2pt) Parameterized surface r.

(b) Compute r_x and r_y.

$$\begin{aligned} r_x &= \mathbf{i} \\ r_y &= \mathbf{j} + \mathbf{k} \end{aligned}$$

Thus, $r_x \times r_y = -\mathbf{j} + \mathbf{k}$. Since the orientation defined on S_2 is upward, this vector is what we want.
Compute $\text{curl}(F)$:

$$\text{curl}(F) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & 2y + z & x^2 \end{vmatrix} = -\mathbf{i} - 2x\mathbf{j} + \mathbf{k}$$

By the definition of the surface integral, we have

$$\begin{aligned} \iint_{S_2} \mathbf{F} d\mathbf{S} &= \iint_D \mathbf{F} \cdot (r_x \times r_y) dA = \iint_D 2x + 1 dA \\ &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}-1}^{\sqrt{1-x^2}-1} 2x dy dx + A(D) \\ &= \int_{-1}^{1} 2x(2\sqrt{1 - x^2}) dx + \pi \\ &= \pi \end{aligned}$$

(2pt) Compute r_x and r_y.

(2pt) Compute $r_x \times r_y$.

(2pt) Compute $\text{curl}(F)$.

(1pt) Any sign to indicate that the student know $\iint_S \mathbf{F} d\mathbf{S} = \iint_D F \cdot (r_x \times r_y) dA$.

(2pt) Give a correct upper limit and lower limit for the double integral

(2pt) Correctly evaluate the integral.

(c) Since the curl(F) is continuous and the surface S_1 and S_2 share the same boundary, we have $\iint_{S_1} \mathbf{F} d\mathbf{F} = \iint_{S_2} \mathbf{F} d\mathbf{F}$ by Stokes' theorem.

(1pt) Mention that curl(F) is continuous or the surface S_1 and S_2 share the same boundary.

(1pt) Mention using Stokes' theorem.

4. Let S be the part of the surface $z = 4 - x^2 - y^2$ in the first octant with upward orientation.

(a) (6%) Parametrize the surface S and find the unit normal vector $\mathbf{n}(x, y, z)$.

(b) (2%) Find a vector field $\mathbf{F}(x, y, z) = \langle 0, 0, f(x, y, z)\rangle$ such that

$$\iint_S \frac{xyz}{\sqrt{4x^2 + 4y^2 + 1}}\, dS = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_S \mathbf{F}\cdot d\mathbf{S}.$$

(c) (6%) Use the Divergence Theorem to show that

$$\iint_S \frac{xyz}{\sqrt{4x^2 + 4y^2 + 1}}\, dS = \iiint_E xy\, dV$$

where E is the solid in the first octant satisfying $0 \le z \le 4 - x^2 - y^2$, $0 \le y \le \sqrt{4 - x^2}$, $0 \le x \le 2$.

(d) (6%) Evaluate either $\iint_S \frac{xyz}{\sqrt{4x^2 + 4y^2 + 1}}\, dS$ or $\iiint_E xy\, dV$ directly.

Solution:

(a) There are multiple possible parametrization choices but the rest of the problem suggests using

$$\mathbf{r}(x, y) = \left\langle x, y, 4 - x^2 - y^2\right\rangle \;,\quad x^2 + y^2 \le 4 \;,\; x, y \ge 0.$$

$$\mathbf{r}_x = \langle 1, 0, -2x\rangle$$

$$\mathbf{r}_y = \langle 0, 1, -2y\rangle$$

$$\mathbf{r}_x \times \mathbf{r}_y = \langle 2x, 2y, 1\rangle$$

$$\mathbf{n}(x, y, z) = \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}\langle 2x, 2y, 1\rangle.$$

(b) For the given vector field $\mathbf{F}(x, y, z) = \langle 0, 0, f(x, y, z)\rangle$,

$$\mathbf{F}\cdot\mathbf{n} = \langle 0, 0, f(x, y, z)\rangle \cdot \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}\langle 2x, 2y, 1\rangle = \frac{f(x, y, z)}{\sqrt{4x^2 + 4y^2 + 1}}.$$

Hence $f(x, y, z) = xyz$ will make the equations true.

(c) To use the Divergence Theorem, we need a closed surface. S is not closed so we need to add other surfaces. Let S_1 be the face of E with $x = 0$, S_2 be the face of E with $y = 0$, and S_3 be the face of E with $z = 0$. Since $f(x, y, z)$ is equal to zero on S_1, S_2, and S_3, the outward flux of $\mathbf{F}$ across them will be zero.

Therefore the equation is true using the given flux in (b), the Divergence Theorem on $\mathbf{F}(x, y, z) = \langle 0, 0, xyz\rangle$, and the solid E (whose boundary is S, S_1, S_2, and S_3). Note that $\text{div}\mathbf{F} = xy$.

(d) We will evaluate both.

$$\iint_S \frac{xyz}{\sqrt{4x^2 + 4y^2 + 1}}\, dS = \iint_{x^2+y^2\le 4, x,y\ge 0} xy(4 - x^2 - y^2)\, dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\sin\theta(4 - r^2)\; r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\cos\theta\sin\theta\, d\theta \int_0^2 4r^3 - r^5\, dr$$

$$= \frac{1}{2}\left(16 - \frac{32}{3}\right) = \frac{8}{3}$$

$$\iiint_E xy\ dV = \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} r^2 \cos\theta \sin\theta\ r\ dz\ dr\ d\theta$$
$$= \int_0^{\pi/2} \cos\theta \sin\theta\ d\theta \int_0^2 4r^3 - r^5\ dr$$
$$= \frac{1}{2}\left(16 - \frac{32}{3}\right) = \frac{8}{3}$$

□

Grading:

- For (a), (2%) for writing a correct $\mathbf{r}$, (2%) for bounds of the parametrization (might be written in later parts of their answer), and (2%) for finding $\mathbf{n}(x, y, z)$. It is okay if they decide to use variables other than x, y, z.
- For (b), all or nothing, no partial credit.
- For (c), (3%) for understanding ∂E and discussing the flux through each face. (3%) for finding div$\mathbf{F}$ and showing understanding of the Divergence Theorem. If student decides to show equality by evaluating both sides, they can get (3%) in (c).
- Computation of the multiple integral uses the simple (-1%) for each minor mistake and (-2%) for each major mistake.

5. Consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}(2x-1)^{2n+1}.$$

(a) (5%) Find its radius of convergence which is denoted by r.

(b) (3%) Does the power series converge at $x = \frac{1}{2} + r$? Explain your answer.

(c) (4%) Find $f(1)$. (Hint: Compare $f(1)$ with the Taylor series of $\arctan x$ at 0.)

Solution:

(a) Let $a_n = \frac{(-1)^n}{(2n+1)3^n}(2x-1)^{2n+1}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2n+1}{2n+3}\frac{|2x-1|^2}{3} \longrightarrow \frac{|2x-1|^2}{3} \quad \text{as} \quad n \to \infty.$$

Hence by the ration test, if $\frac{|2x-1|^2}{3} < 1$ i.e. $|x - \frac{1}{2}| < \frac{\sqrt{3}}{2}$, the power series converges. If $|x - \frac{1}{2}| > \frac{\sqrt{3}}{2}$, the power series diverges. Hence the radius of convergence is $\frac{\sqrt{3}}{2}$.

(2 pts for $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|2x-1|^2}{3}$.

1 pt for stating that the series converges if $\frac{|2x-1|^2}{3} < 1$ and it diverges if $\frac{|2x-1|^2}{3} > 1$.

2 pts for the radius of convergence $\frac{\sqrt{3}}{2}$.)

(b) When $x = \frac{1}{2} + r$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}(1+\sqrt{3}-1)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}\sqrt{3}.$$

The above series is an alternating series. Moreover, $\{\frac{\sqrt{3}}{2n+1}\}$ is decreasing and $\lim_{n\to\infty}\frac{\sqrt{3}}{2n+1} = 0$. Hence by the alternating series test, the power series converges when $x = \frac{1}{2} + r$.

(1 pt for plugging $x = \frac{1}{2} + r$ into the power series.

1 pt for stating that $\{\frac{\sqrt{3}}{2n+1}\}$ is decreasing and $\lim_{n\to\infty}\frac{\sqrt{3}}{2n+1} = 0$.

1 pt for deriving the convergence of the power series by the alternating series test.)

(c)

$$f(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

Note that

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}.$$

Hence

$$\arctan(\frac{1}{\sqrt{3}}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(\frac{1}{\sqrt{3}})^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n\sqrt{3}}.$$

Therefore $f(1) = \sqrt{3}\arctan(\frac{1}{\sqrt{3}}) = \frac{\sqrt{3}}{6}\pi$.

(1 pt for $f(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$. 1 pt for the Maclaurin series of $\arctan x$.

1.5 pts for $f(1) = \sqrt{3}\arctan(\frac{1}{\sqrt{3}})$. 0.5 pt for $f(1) = \frac{\sqrt{3}}{6}\pi$.)

6. Let $f(x) = \int_0^{2x} \cos(t^2)\, dt$.

(a) (3%) Write down the Taylor series of $g(t) = \cos(t^2)$ centered at 0.

(b) (4%) Derive the Taylor series of $f(x)$ centered at 0.

(c) (3%) Find $f^{(113)}(0)$.

(d) (3%) Let $T_5(x)$ be the 5th-degree Taylor polynomial of $f(x)$ centered at 0. Compute $T_5(0.1)$.

(e) (4%) We can use $T_5(0.1)$ to approximate $f(0.1)$. Give an upper bound for $|f(0.1) - T_5(0.1)|$.

Solution:

(a)

$$g(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!}(t^2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} t^{4n}.$$

(1 pt for the Maclaurin series of $\cos x$,
2 pts for plugging t^2 into the Maclaurin series of $\cos x$.)

(b) By the term-by-term integration theorem,

$$f(x) = \int_0^{2x} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} t^{4n}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \frac{2^{4n+1}}{4n+1} x^{4n+1}.$$

(1 pt for replacing $\cos(t^2)$ by its Maclaurin series.
1 pt for trying integrating term-by-term. 2 pts for the final answer.)

(c) We know that the Maclaurin series of $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

On the other hand, we have shown that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \frac{2^{4n+1}}{4n+1} x^{4n+1}.$$

Compare the coefficients before x^{113} in these two power series. We obtain that

$$\frac{f^{(113)}(0)}{113!} = \frac{(-1)^{28}}{56!} \frac{2^{113}}{113}. \quad \Longrightarrow \quad f^{(113)}(0) = 2^{113} \frac{112!}{56!}.$$

(1 pt for stating that the coefficient before x^{113} in the Maclaurin series is $\frac{f^{(113)}(0)}{113!}$.
1 pt for finding that the coefficient before x^{113} in the Maclaurin series is $\frac{(-1)^{28}}{56!} \frac{2^{113}}{113}$.
1 pt for $f^{(113)}(0)$.)

(d) By part (b), $T_5(x) = 2x - \frac{16}{5}x^5$.
$T_5(0.1) = 0.2 - \frac{16}{500000} = 0.2 - \frac{1}{2 \times 5^6}$.
(1 pt for $T_5(x)$. 2 pts for $T_5(0.1)$.)

(e) Observe that $f(0.1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \frac{2^{4n+1}}{4n+1} (0.1)^{4n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!(4n+1)5^{4n+1}}$ is an alternating series.
Moreover, the sequence $\{\frac{1}{2n!(4n+1)5^{4n+1}}\}$ is decreasing and $\lim_{n\to\infty} \frac{1}{2n!(4n+1)5^{4n+1}} = 0$. Hence, by the alternating series estimation theorem,

$$|f(0.1) - T_5(0.1)| \le \frac{1}{4! \times 9 \times 5^9}.$$

(1 pt for observing that $f(0.1)$ is the sum of an alternating series.
1 pt for stating that the sequence $\{\frac{1}{2n!(4n+1)5^{4n+1}}\}$ is decreasing and $\lim_{n\to\infty} \frac{1}{2n!(4n+1)5^{4n+1}} = 0$.
1 pt for applying the alternating series estimation theorem.
1 pt for $|f(0.1) - T_5(0.1)| \le \frac{1}{4! \times 9 \times 5^9}$.)