112 模組01-04班 微積分4 期考解答和評分標準

1. Evaluate the following integrals.

   (a) (4%) $\int_C \frac{1}{x^2+y^2+z^2}\,ds$ where C is the helix $x=\cos t, y=\sin t, z=t$ $(0\le t\le T)$.

   (b) (6%) $\iint_S (x^2+y^2+z^2)\,dS$ where S is the portion of the plane $z=x+1$ that lies inside the cylinder $x^2+y^2=1$.

   (c) (6%) $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z)=\frac{1}{(x^2+y^2+z^2)^{3/2}}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$ and S is part of the sphere $x^2+y^2+z^2=1$ inside the cone $z=\sqrt{x^2+y^2}$, oriented away from the origin.

Solution:

(a) We have (+2)

$$ds=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dt=\sqrt{(-\sin t)^2+(\cos t)^2+1^2}\,dt=\sqrt{2}\,dt.$$

Hence

$$\begin{aligned}\int_C \frac{1}{x^2+y^2+z^2}ds &= \int_0^T \frac{1}{\cos^2 t+\sin^2 t+t^2}\sqrt{2}\,dt \quad (+1)\\ &= \sqrt{2}\tan^{-1} t\Big|_0^T\\ &= \sqrt{2}\tan^{-1} T \quad (+1).\end{aligned}$$

[Write ds in terms of t correctly: (+2);
turn the line integral into an integral for variable t correctly: (+1);
obtain the correct answer: (+1).]

(b) Let D be the unit disk $x^2+y^2\le 1$ on the xy-plane. The surface S is given by the parametrization

$$\mathbf{r}(x,y)=\langle x,y,x+1\rangle,\quad (x,y)\in D \quad (+3)$$

with the normal

$$\mathbf{r}_x\times\mathbf{r}_y=\langle -1,0,1\rangle$$

having length $\sqrt{2}$. Thus

$$\iint_S (x^2+y^2+z^2)\,dS=\iint_D (x^2+y^2+(x+1)^2)\cdot\sqrt{2}\,dA \quad (+2).$$

Under the polar coordinates

$$(x,y)=(r\cos\theta, r\sin\theta),\quad 0\le r\le 1, 0\le\theta\le 2\pi,$$

the latter integral equals

$$\sqrt{2}\int_0^{2\pi}\int_0^1 (r^2+(r\cos\theta+1)^2)r\,dr\,d\theta=\frac{7\sqrt{2}}{4}\pi \quad (+1).$$

[Find a correct parametrization compatible with orientation, including the domain for the parameters: (+3);
turn correctly the surfae integral into an integral for the parameters: (+2);
obtain the correct answer: (+1);
incorrect order of the parameter: (−1);
incorrect domain for the parameters: (−1).]

(c) Under the spherical coordinates (ρ,ϕ,θ), the sphere $x^2+y^2+z^2=1$ is given by $\rho=1$; the cone $z=\sqrt{x^2+y^2}$ becomes $\rho\cos\phi=\rho\sin\phi$, that is, $\phi=\pi/4$. Thus the oriented surface S is parametrized by (+3)

$$\mathbf{r}(\phi,\theta)=\langle\sin\phi\cos\theta,\sin\phi\sin\theta,\cos\phi\rangle,\quad 0\le\phi\le\frac{\pi}{4}, 0\le\theta\le 2\pi.$$

One computes the normal

$$\mathbf{r}_\phi\times\mathbf{r}_\theta=\sin\phi\cdot\mathbf{r},$$

which has length $\sin\phi$. One has

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_S \mathbf{r}\cdot\mathbf{r}\,dS = \iint_S dS \\ &= \int_0^{2\pi}\int_0^{\pi/4} \sin\phi\,d\phi d\theta \quad (+2) \\ &= (2-\sqrt{2})\pi \quad (+1).\end{aligned}$$

[Find a correct parametrization compatible with orientation, including the domain for the parameters: (+3);
turn correctly the flux integral into an integral for the parameters: (+2);
obtain the correct answer: (+1);
incorrect order of the parameter: (−1);
incorrect domain for the parameters: (−1).]

2. Consider two vector fields

$$\mathbf{F}(x,y) = -\frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j} \text{ and } \mathbf{G}(x,y) = \frac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{j}.$$

(a) (i) (4%) Find $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is the positively oriented circle $x^2+y^2=1$.

(ii) (2%) Is $\mathbf{F}$ conservative on $\mathbb{R}^2 \smallsetminus \{(0,0)\}$? Explain your answer.

(iii) (6%) Let C' be the polar curve $r = 3+\cos\theta$, oriented positively. Find $\oint_{C'} \mathbf{F}\cdot d\mathbf{r}$.

(b) (i) (2%) Is $\mathbf{G}$ conservative on $\mathbb{R}^2 \smallsetminus \{(0,0)\}$? Explain your answer.

(ii) (2%) Let L be the line segment from $(2,0)$ to $(3,4)$. Find $\int_L \mathbf{G}\cdot d\mathbf{r}$.

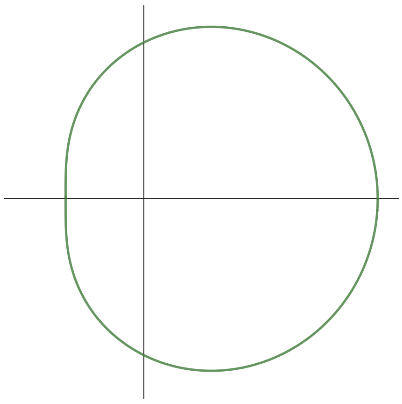

Figure. The polar curve $r = 3+\cos\theta$

Solution:

(a) (i) Parameterize C as $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = 2\pi$$

- (1%) Correct parametrization of C, including the domain of parameters
- (2%) Correct definition of a line integral
- (1%) Correct answer

(ii) (Method 1) By (a), $\oint_C \mathbf{F}\cdot d\mathbf{r} = 2\pi/ \neq 0$ and C is a curve on $\mathbb{R}^2 \smallsetminus \{(0,0)\}$. Therefore, $\mathbf{F}$ is not conservative on $\mathbb{R}^2 \smallsetminus \{(0,0)\}$.

(Method 2) Since $Q_x - P_y = \frac{1}{\sqrt{x^2+y^2}} \neq 0$, we conclude that $\mathbf{F}$ is not conservative on any path-connected open subset of $\mathbb{R}^2$, including $\mathbb{R}^2 \smallsetminus \{(0,0)\}$.

- (1%) Correct justification (Note that saying 'No' without any valid justification will receive no credits).
- (1%) Overall coherency of the argument (no sloppiness).

(ii) Let D be the region enclosed by C and C' (which does not enclose the origin). Now apply the Generalized Green's Theorem,

$$\oint_{C'} \mathbf{F}\cdot d\mathbf{r} - \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \frac{1}{\sqrt{x^2+y^2}} dA = \int_0^{2\pi}\int_1^{3+\cos\theta} \frac{1}{r}\cdot r\,dr\,d\theta = \int_0^{2\pi} (2+\cos\theta)\,d\theta = 4\pi$$

Thus, $\oint_{C'} \mathbf{F}\cdot d\mathbf{r} = 4\pi + 2\pi = 6\pi$

- (1%) Consider to apply Generalized Green's Theorem
- (2%) Convert the difference of the two line integrals into a correct double integral (including that the integrand has to be correct)
- (2%) Correct computation of the double integral
- (1%) Correct answer

(b) (i) Yes. $g(x,y) = \sqrt{x^2 + y^2}$ is a scalar potential function of $\mathbf{G}$.

(ii) By FTC for line integrals, we have

$$\int_L \mathbf{G} \cdot d\mathbf{r} = g(3,4) - g(2,0) = 5 - 2 = 3.$$

Grading scheme for (b)(i),(ii)

- (1%) Correct justification for conservativesness (Just saying Yes without any valid justification will receive no credits)
- (1%) Correct scalar potential function
- (1%) Statement of FTC for line integrals
- (1%) Correct answer

3. Let S be the part of the cylinder $y^2 + z^2 = 1$ in the first octant between the planes $x = 0$ and $x + y = 1$, oriented upward. Consider the vector field

$$\mathbf{F}(x, y, z) = (e^x + y)\mathbf{i} + \ln(1 + x)\mathbf{j} + y\mathbf{k}.$$

(a) (3%) Parametrize the surface S.

(b) (7%) Compute, directly, $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$.

(c) Let C be the part of the circle $y^2 + z^2 = 1$, $x = 0$ in the first octant from $(0, 1, 0)$ to $(0, 0, 1)$. Let L be the line segment from $(0, 0, 1)$ to $(1, 0, 1)$.

i. (4%) Compute, directly, $\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\int_L \mathbf{F} \cdot d\mathbf{r}$.

ii. (2%) Let C_1 be the curve of intersection of $y^2 + z^2 = 1$ and $x + y = 1$ in the first octant from $(1, 0, 1)$ to $(0, 1, 0)$. Find $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ by Stokes' theorem.

Solution:

(a)

$$S: \mathbf{r}(\theta, x) = (x, \cos\theta, \sin\theta), \qquad 0 \le \theta \le \frac{\pi}{2},\ 0 \le x \le 1 - \cos\theta.$$

(1 pt for $\mathbf{r}(\theta, x)$. 2 pts for ranges of x and θ.)

(b)

$$\text{curl } \mathbf{F} = \mathbf{i} + 0\mathbf{j} + (\frac{1}{x+1} - 1)\mathbf{k}. \qquad (1 \text{ pt})$$

$$\mathbf{r}_\theta \times \mathbf{r}_x = (0, \cos\theta, \sin\theta) \quad \text{which is upward. (1 pt)}$$

Hence

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_0^{\frac{\pi}{2}} \int_0^{1-\cos\theta} \text{curl } \mathbf{F}(\mathbf{r}(\theta, x)) \cdot \mathbf{r}_\theta \times \mathbf{r}_x \, dx d\theta = \int_0^{\frac{\pi}{2}} \int_0^{1-\cos\theta} \frac{\sin\theta}{1+x} - \sin\theta \, dx d\theta \qquad (2 \text{ pts})$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \ln(2 - \cos\theta) - \sin\theta(1 - \cos\theta) \, d\theta \overset{u=2-\cos\theta}{=} \int_1^2 \ln(u) - u + 1 \, du = 2\ln 2 - \frac{3}{2}. \qquad (3 \text{ pts})$$

(Students can get partial credits for the last integration if they only make minor mistakes.)

(c) (i)

$$C: \mathbf{r}(\theta) = (0, \cos\theta, \sin\theta), \quad 0 \le \theta \le \frac{\pi}{2}. \qquad (0.5 \text{ pt})$$

Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\frac{\pi}{2}} (1 + \cos\theta, 0, \cos\theta) \cdot (0, -\sin\theta, \cos\theta) \, d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \frac{\pi}{4}.$$

(1 pt for writing the line integral as an integral with respect to θ. 1 pt for the final answer.)

$$L: \mathbf{r}(t) = (t, 0, 1), \quad 0 \le t \le 1. \qquad (0.5 \text{ pt})$$

Hence

$$\int_L \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (e^t, \ln(1 + t), 0) \cdot (1, 0, 0) \, dt = \int_0^1 e^t \, dt = e - 1. \quad (1 \text{ pt})$$

(ii) By Stokes' Theorem,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{F} \cdot d\mathbf{r} + \int_L \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}. \qquad (1 \text{ pt})$$

Therefore,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} - \int_C \mathbf{F} \cdot d\mathbf{r} - \int_L \mathbf{F} \cdot d\mathbf{r} = 2\ln 2 - \frac{1}{2} - \frac{\pi}{4} - e. \qquad (1 \text{ pt})$$

4. (12%) Let S be the part of the surface $x^2 + (y+1) \cdot z^2 = 1$ for which $0 \le y \le 3$.

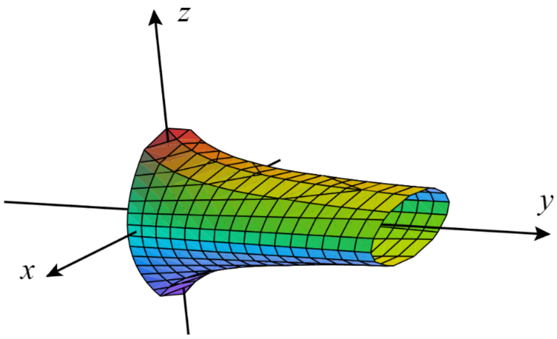

Figure. The surface S

You are given that the volume of the solid enclosed by S and the planes $y = 0$ and $y = 3$ equals 2π. By an appropriate use of the Divergence Theorem, compute the flux of the vector field

$$\mathbf{F}(x,y,z) = (4x + y^2)\mathbf{i} + (x^2 + y^2 - y)\mathbf{j} + (x^2 - 2yz)\mathbf{k}.$$

across S, oriented away from the origin.

Solution:

Let $D_1 = \{(x,y,z) : x^2 + z^2 \le 1, y = 0\}$ and $D_2 = \{(x,y,z) : x^2 + 4z^2 \le 1, y = 3\}$, oriented to the negative and to the positive y-axis respectively. By Divergence Theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_U \operatorname{div}\mathbf{F} dV - \iint_{D_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{D_2} \mathbf{F} \cdot d\mathbf{S}.$$

Now

$$\iint_{D_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{D_1} x^2 + y^2 - y dA = -\int_0^{2\pi} \int_0^1 r^3 \cos^2\theta \, dr d\theta = -\frac{\pi}{4},$$

$$\iint_{D_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{D_2} x^2 + y^2 - y dA = \frac{1}{2} \int_0^{2\pi} \int_0^1 (r^2 \cos^2\theta + 6) r \, dr d\theta = \frac{\pi}{8} + 3\pi = \frac{25\pi}{8}.$$

On the other hand, since $\operatorname{div}\mathbf{F} = 3$, we have

$$\iiint_U \operatorname{div}\mathbf{F} \, dV = 3 \cdot \text{Volume}(U) = 6\pi.$$

Hence, $\iint_S \mathbf{F} \cdot d\mathbf{S} = 6\pi + \frac{\pi}{4} - \frac{25\pi}{8} = \frac{25\pi}{8}$.

- (1%) Correctly defining the surface D_1 (disc), make sure candidates are not confusing curves with surfaces.
- (1%) Correct parametrization of D_1
- (2%) Correct flux of $\mathbf{F}$ across D_1
- (1%) Correctly defining the surface D_2 (elliptical disc), make sure candidates are not confusing curves with surfaces.
- (1%) Correct parametrization of D_2
- (2%) Correct flux of $\mathbf{F}$ across D_2
- (1%) Correct div($\mathbf{F}$)
- (2%) Correct statement of Divergence Theorem (Be aware of whether the orientations of D_1, D_2 are chosen suitably)
- (1%) Correct answer

5. Consider the power series $f(x) = \sum_{n=0}^{\infty} \frac{3^{2n+1}}{2n+3} \cdot x^{2n+5}$.

(a) (4%) Find the radius of convergence of $f(x)$.

(b) (6%) Express $f(x)$ as an elementary function.

Solution:

(a) Let $a_n = \frac{3^{2n+1}}{2n+3} x^{2n+5}$.Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{9(2n+3)|x^2|}{2n+5} = \frac{9(2+3/n)|x^2|}{2+5/n} \longrightarrow |9x^2| \text{ as } n \longrightarrow \infty.$$

By the ratio test, $f(x)$ converges if $|9x^2| < 1$ and diverges if $|9x^2| > 1$. Hence $R = \frac{1}{3}$. Marking scheme:

- (3pts) For correctly calculating the limit of $|a_{n+1}/a_n|$.
- (1pt) For finding the radius of convergence using the ratio test.

Alternative method(root test):

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{3^{2n+1}}{2n+3} |x|^{2n+5}} = |9x^2| \left(\frac{3|x|^5}{2n+3} \right)^{\frac{1}{n}}.$$

By L'Hôspital's rule,

$$\lim_{x \to \infty} \frac{\ln(2x+3)}{x} = \lim_{x \to \infty} \frac{\frac{2}{2x+3}}{1} = 0.$$

Hence

$$(2n+3)^{\frac{1}{n}} = e^{\frac{1}{n} \ln(2n+3)} \longrightarrow e^0 = 1 \text{ as } n \longrightarrow \infty,$$

since e^x is continuous. As a result,

$$\left(\frac{3|x|^5}{2n+3} \right)^{\frac{1}{n}} \longrightarrow 1 \text{ as } n \longrightarrow \infty, \text{ if } x \neq 0, \text{ and}$$
$$\longrightarrow 0 \text{ as } n \longrightarrow \infty, \text{ if } x = 0.$$

In both cases, we have

$$\sqrt[n]{|a_n|} \longrightarrow |9x^2| \text{ as } n \to \infty.$$

By the root test, $f(x)$ converges if $|9x^2| < 1$ and diverges if $|9x^2| > 1$. Hence $R = \frac{1}{3}$.

Marking scheme:

- (3pts) For correctly calculating the limit of $\sqrt[n]{|a_n|}$ with a complete argument. Partial credit:
  - (2pts) The limit is correct but the argument is not complete.
  - (1pt) The limit is correct but the argument is poorly written or there is no argument.
- (1pt) For finding the radius of convergence using the root test.

(b) We have $f(x) = x^2 \sum_{n=0}^{\infty} \frac{3^{2n+1}}{2n+3} x^{2n+3}$. Let $g(x) = \sum_{n=0}^{\infty} \frac{3^{2n+1}}{2n+3} x^{2n+3}$, and consider the term-by-term differentiation of $g(x)$:

$$g'(x) = \sum_{n=0}^{\infty} 3^{2n+1} x^{2n+2} = 3x^2 \sum_{n=0}^{\infty} (9x^2)^n = \frac{3x^2}{1-9x^2}.$$

Hence

$$g(x) = \int \frac{3x^2}{1-9x^2} dx = \int [\frac{-1}{3} + \frac{1}{6}(\frac{1}{1+3x} + \frac{1}{1-3x})] \, dx$$
$$= \frac{-x}{3} + \frac{1}{18} \ln\left(\frac{1+3x}{1-3x}\right) + C.$$

Note that $\frac{1+3x}{1-3x} > 0$ when $|x| < \frac{1}{3}$. Evaluating at $x = 0$, we get $C = 0$ and

$$f(x) = \frac{-x^3}{3} + \frac{x^2}{18} \ln\left(\frac{1+3x}{1-3x}\right).$$

Marking scheme:

- (2pts) For the expression $f(x) = x^2 g(x)$, and for expressing $g'(x)$ as an elementary function through the term-by-term differentiation.
  Partial credit:
  - (1pt) (when failing to find $g'(x)$) If a correct term-by-term differentiation is carried out for any power series.
- (3pts) For finding $g(x)$ by integrating $g'(x)$.
  Partial credit:
  - (1pt-2pts) There are minor mistakes in the calculation.
- (1pt) Thus, find the expression of $f(x)$ in terms of elementary functions.

6. Consider the function $f(x) = \dfrac{(1+2x^5)^e - 1}{x^5}$.

(a) (4%) Find the power series $\sum_{n=0}^{\infty} c_n x^n$ centered at $x = 0$ such that

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \text{ for } 0 < |x| < R.$$

Determine the greatest possible value of R.

(b) (2%) Hence find the limit $L = \lim_{x\to 0} f(x)$.

In the remaining parts, we extend the domain of f by setting $f(0) = L$.

(c) (4%) Find $f^{(10)}(0)$. Don't just express your answer in terms of a binomial coefficient for this part.

(d) (2%) Let $F(x) = \int_0^x f(t)\,dt$. Find the Maclaurin series of $F(x)$.

(e) (5%) Hence, express $F(0.5) = \int_0^{0.5} f(t)\,dt$ as an infinite series $b = \sum_{n=0}^{\infty} b_n$ where each b_n is non-zero. Prove that

$$|b - (b_0 + b_1)| < 2^{-8}.$$

Solution:

(a) Since $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, we have

$$\frac{(1+2x^5)^e - 1}{x^5} = \frac{\sum_{n=1}^{\infty} \binom{e}{n}(2x^5)^n}{x^5} = \sum_{n=0}^{\infty} \binom{e}{n+1} 2^{n+1} x^{5n}.$$

It converges when $|2x^5| < 1$ and diverges when $|2x^5| > 1$. Hence $R = \dfrac{1}{\sqrt[5]{2}}$.

Marking scheme:

- (2pts) For correctly finding the Maclaurin series of $f(x)$. Index shifting is allowed. Partial credit:
  - (1pt) The answer is not correct but the correct expression for the binomial series can be found.
- (2pts) For finding the radius of convergence. One can use the fact that the radius of convergence of the binomial series is 1 without proof.
  Partial credit:
  - (1pt) The inequalities are correctly written but somehow the final answer is not correct.

(b) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \sum c_n x^n = c_0 = 2e$.

Marking scheme:

- Partial credit (when the answer is not correct):
  - (1pt) For knowing that $L = c_0$. This point is awarded even if the series calculated in (a) is wrong.

(c) $f^{(10)}(0) = 10!c_{10} = \dfrac{4 \cdot 10!e(e-1)(e-2)}{3}$.

Marking scheme:

- No need to expand the factorial.
- Partial credit (when the answer is not correct):
  - (1pt) For knowing that $f^{(10)}(0) = 10!c_{10}$.
  - (1pt) For correctly expanding the binomial coefficient. It can be any $\binom{e}{k}$, $k \neq 1$.

  These points are awarded even if the series calculated in (a) is wrong.

(d)

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} \binom{e}{n+1} 2^{n+1} x^{5n}\,dx = \sum_{n=0}^{\infty} \binom{e}{n+1} \frac{2^{n+1}x^{5n+1}}{5n+1} + C.$$

Evaluating at $x = 0$, we get $C = 0$ and

$$F(x) = \sum_{n=0}^{\infty} \binom{e}{n+1} \frac{2^{n+1}x^{5n+1}}{5n+1}.$$

Marking scheme:

- Partial credit:
  - (1pt) The whole process is correct but the answer is wrong because of (a).

(e)

$$\begin{aligned} F(0.5) = \int_0^{0.5} f(t)\,dt &= \sum_{n=0}^{\infty} \binom{e}{n+1} \frac{2^{n+1}}{2^{5n+1}(5n+1)} \\ &= \sum_{n=0}^{\infty} \binom{e}{n+1} \frac{1}{2^{4n}(5n+1)}. \end{aligned}$$

Let $b_n = \binom{e}{n+1} \frac{1}{2^{4n}(5n+1)}$. Then we have

$$b - (b_0 + b_1) = \sum_{n=2}^{\infty} b_n$$

and $\sum_{n=2}^{\infty} b_n$ is an alternating series. Furthermore, we have

$$|b_{n+1}/b_n| = \frac{|e-n-1|}{2^4(n+2)} \left| \frac{5n+1}{5n+6} \right| < 1 \text{ for all } n \geq 0.$$

Hence, by the alternating series estimation theorem,

$$|b - (b_0 + b_1)| < |b_2| = b_2 = \frac{1}{2^8} \cdot \binom{e}{3} \frac{1}{11} = \frac{1}{2^8} \frac{e(e-1)(e-2)}{66} < 2^{-8}.$$

Marking scheme:

- For this part, no point is awarded if the series calculated in (a) is already wrong.
- (1pt) For expressing $F(0.5)$ as an infinite series.
- (2pts) For checking the condition $|b_{n+1}| < |b_n|$.
- (2pts) For correctly applying the alternating series estimation theorem.
  Partial credit:
  - (1pt) If the student directly write down the estimation without mentioning that $\sum b_n$ is alternating starting from $n = 2$.

7. Let $f(x,y,z) = \sqrt[3]{1+(x-1)^2+yz}$.

(a) (2%) Show that $(1,0,0)$ is a critical point of $f(x,y,z)$.

(b) (2%) Write down $T_1(w)$, the first degree Taylor polynomial of $g(w) = \sqrt[3]{1+w}$ at $w = 0$.

(c) (4%) Recall the *Lagrange's form of remainder* :

> Suppose that $T_n(x)$ is the nth-degree Taylor polynomial of $g(x)$ at $x = 0$ and $g^{(n+1)}(x)$ is continuous. In this case, the Lagrange's form of remainder is
>
> $$g(x) - T_n(x) = \frac{g^{(n+1)}(c)}{(n+1)!}x^{n+1} \qquad \text{for some } c \text{ between 0 and } x.$$

Use it to prove that $|\sqrt[3]{1+w} - T_1(w)| < \frac{1}{8}w^2$ for $|w| \le 0.01$.

(d) (5%) By considering the line $\mathbf{r}(t) = \langle 1+t, t, -2t\rangle$ and using (c), estimate the value of $f(1.1, 0.1, -0.2)$. Give an upper bound for the error in your estimation.

Solution:

(a)

$$f_x = \frac{2(x-1)}{3(1+(x-1)^2+yz)^{\frac{2}{3}}}, \quad f_y = \frac{z}{3(1+(x-1)^2+yz)^{\frac{2}{3}}}, \quad f_z = \frac{y}{3(1+(x-1)^2+yz)^{\frac{2}{3}}}.$$

Hence $\nabla f(1,0,0) = \mathbf{0}$ and $(1,0,0)$ is a critical point of f.
(1 pt for computing partial derivatives of f. 1 pt for stating that $\nabla f(1,0,0) = \mathbf{0}$.)

(b) $T_1(w) = 1 + \frac{1}{3}w$. (2 pts)

(c) $|\sqrt[3]{1+w} - T_1(w)| = \frac{|g''(c)|}{2}w^2$ for some c between 0 and w. Since $g''(c) = -\frac{2}{9(1+c)^{\frac{5}{3}}}$ and $|c| < 0.01$, we know that

$$|g''(c)| < \frac{2}{9(0.99)^{\frac{5}{3}}} < \frac{2}{9(0.99)^2} < \frac{2}{8} = \frac{1}{4}.$$

Hence

$$|\sqrt[3]{1+w} - T_1(w)| = \frac{|g''(c)|}{2}w^2 < \frac{1}{8}w^2.$$

(1 pt for $|\sqrt[3]{1+w} - T_1(w)| = \frac{|g''(c)|}{2}w^2$. 1 pt for computing $g''(c)$. 2 pts for estimating $|g''(c)|$.)

(d)

$$f(\mathbf{r}(t)) = \sqrt[3]{1+t^2-2t^2} = \sqrt[3]{1-t^2} = g(-t^2).$$

Hence $f(1.1, 0.1, -0.2) = f(\mathbf{r}(0.1)) = g(-0.01)$. From part (b), we know that

$$g(-0.01) \approx T_1(-0.01) = 1 - \frac{1}{300} = \frac{299}{300}.$$

Moreover, part (c) says that $|g(-0.01) - T_1(-0.01)| < \frac{1}{8}(0.01)^2 = \frac{1}{80000}$.
(1 pt for $f(1.1, 0.1, -0.2) = g(-0.01)$, 2 pts for $g(-0.01) \approx T_1(-0.01) = \frac{299}{300}$. 2 pts for the upper bound of the error, 1/80000.)