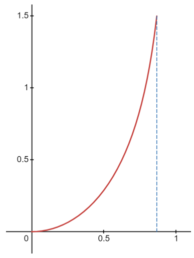
1. Consider a parametric curve $C:\ x(t) = \sin t,\ \ y(t) = \sin t \tan t,\ \ \text{for } 0 \le t \le \frac{\pi}{3}.$



(a) (5%) Find the slope of the tangent line to C, $\frac{dy}{dx}$, at $t = \frac{\pi}{4}$.

(b) (5%) Compute the area of the region under C, $0 \le t \le \frac{\pi}{3}$, and above the x−axis.

Solution:

(a) $\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)}$, (1 pt)

where $y'(\theta) = \cos\theta\tan\theta + \sin\theta\sec^2\theta = \sin\theta + \sin\theta\sec^2\theta$ (2 pts)

and $x'(\theta) = \cos\theta$. (1 pt)

At $\theta = \pi/4$, $\frac{dy}{dx} = \frac{y'(\frac{\pi}{4})}{x'(\frac{\pi}{4})} = \frac{\sin(\frac{\pi}{4}) + \sin(\frac{\pi}{4})\sec^2(\frac{\pi}{4})}{\cos(\frac{\pi}{4})} = 3.$ (1 pt)

(b)

$$A = \int_0^{\frac{\pi}{3}} y(\theta)x'(\theta)\, d\theta \qquad \text{(2 pts)}$$

$$= \int_0^{\frac{\pi}{3}} \sin\theta\tan\theta\cos\theta\, d\theta = \int_0^{\frac{\pi}{3}} \sin^2\theta\, d\theta \qquad \text{(1 pt)}$$

$$= \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\Bigg|_{\theta=0}^{\theta=\frac{\pi}{3}} \qquad \text{(1 pt)}$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}. \qquad \text{(1 pt)}$$

2. Let $f(x,y)$ be a differentiable function. Suppose you are given the following table.

$f(2,3)=10$	$f_x(2,3)=-2$	$f_y(2,3)=3$	$f(4,5)=-11$	$f_x(4,5)=6$	$f_y(4,5)=4$

(a) (4%) Let $C: f(x,y)=10$ be the level curve that passes through $(2,3)$. Find the tangent line to the level curve C at $(x,y)=(2,3)$.

(b) (4%) Let $\mathcal{S}$ be the surface $z=f(x,y)$. Find the tangent plane to the surface $\mathcal{S}$ at $(x,y,z)=(4,5,f(4,5))$.

(c) (6%) Let $g(u,v)=f(uv-2,\ v^2-u^2)$. Compute $\dfrac{\partial g}{\partial u}$ and $\dfrac{\partial g}{\partial v}$ at $(u,v)=(2,3)$.

Solution:

(a) (2M) By Implicit Function Theorem, we have $\left.\dfrac{dy}{dx}\right|_{(2,3)}=-\dfrac{f_x(2,3)}{f_y(2,3)}$.

(1M) This gives $\left.\dfrac{dy}{dx}\right|_{(2,3)}=-\dfrac{-2}{3}=\dfrac{2}{3}$.

(1M) Therefore, the equation of tangent line is $y-3=\dfrac{2}{3}(x-2)$.

Grading Scheme.

- (2M) For obtaining a formula for $\dfrac{dy}{dx}$ (Including, for example, memorizing the formula for tangent line)
- (1M) For the correct slope
- (1M) For the correct equation

(b) (Method 1)

(1M) The equation of tangent plane of $z=f(x,y)$ is

$$z=f(4,5)+f_x(4,5)(x-4)+f_y(4,5)(y-5)$$

(1+1+1M) which equals $z=-11+6(x-4)+4(y-5)$.

(Method 2)

A normal of $z=f(x,y)$ (at an arbitrary point) is given by $\langle -f_x, -f_y, 1\rangle$.

(1+1+1M) In particular, at $(4,5,-11)$, this becomes $\langle -6,-4,1\rangle$.

(1M) Hence, the equation of tangent plane at $(4,5,-11)$ equals

$$\langle -6,-4,1\rangle\cdot\langle x-4,y-5,z+11\rangle=0\Rightarrow -6(x-4)-4(y-5)+(z+11)=0.$$

Grading Scheme.

- (1+1+1M) For the coefficents of x, of y and of constant term
- (1M) Knowing the method of finding a tangent plane

(c) By the chain rule

$$\boxed{\text{(1M)}}\quad \begin{aligned} g_u &= f_x(uv-2,v^2-u^2)\cdot x_u+f_y(uv-2,v^2-u^2)\cdot y_u\\ &= f_x(uv-2,v^2-u^2)\cdot v+f_y(uv-2,v^2-u^2)\cdot(-2u)\\ g_v &= f_x(uv-2,v^2-u^2)\cdot x_v+f_y(uv-2,v^2-u^2)\cdot y_v\\ &= f_x(uv-2,v^2-u^2)\cdot u+f_y(uv-2,v^2-u^2)\cdot(2v)\end{aligned}$$

(3M) Therefore, $g_u(2,3)=f_x(4,5)\cdot 3+f_y(4,5)\cdot(-4)$ and $g_v(2,3)=f_x(4,5)\cdot 2+f_y(4,5)\cdot(6)$.

(1+1M) Thus, $g_u(2,3)=6\cdot 3+4\cdot(-4)=2$ and $g_v(2,3)=6\cdot 2+4\cdot(6)=36$.

Grading Scheme.

- (1M) For demonstrating some (correct) knowledge about chain rule.
- (3M) For writing down the <u>precise</u> equality for either $g_u(2,3)$ <u>or</u> $g_v(2,3)$.
- (1+1M) For the correct answers.

3. Suppose that $f(x, y)$ is a differentiable real-valued function on $\mathbb{R}^2$ having the following three properties:

- $f(0,0) = 2$.
- $f_x(x,0) = e^x$ for all $x \in \mathbb{R}$.
- The directional derivative of $f(x, y)$ in the direction $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is equal to 0 at every point of $\mathbb{R}^2$.

(a) (4%) Find $\nabla f(0,0)$.

(b) (4%) For fixed constants a and b, let $g(t) = f(a+t, b+t)$ for all $t \in \mathbb{R}$. Show that $g(t)$ is a constant function.

(c) (3%) Find the value of $f(1,2)$.

Solution:

(a) By assumption, $0 = D_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})} f(0,0) = \frac{1}{\sqrt{2}} f_x(0,0) + \frac{1}{\sqrt{2}} f_y(0,0)$ (1%), so $f_y(0,0) = -f_x(0,0)$ (1%). Since $f_x(0,0) = e^0 = 1$, we conclude that $\nabla f(0,0) = (f_x(0,0), f_y(0,0))$ (1%) $= (1, -1)$ (1%). □

(b) *Method 1:* Since $D_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})} f = 0$ everywhere, f is constant on each line of direction $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, so that for fixed constants a and b, $g(t) = f(a+t, b+t)$ is a constant function of t. (4%) □

Method 2: By assumption, $0 = D_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})} f = \frac{1}{\sqrt{2}} f_x + \frac{1}{\sqrt{2}} f_y$ everywhere (2%), so $f_x + f_y = 0$ everywhere; thus $g'(t) = f_x(a+t, b+t) + f_y(a+t, b+t) = 0$ for all $t \in \mathbb{R}$ (1%), and hence $g(t)$ is a constant function (1%). □

(c) By (b), we have $f(1,2) = f(-1,0)$ (1%). Since

$$f(-1,0) = f(0,0) + \int_0^{-1} f_x(x,0)\,dx \text{ (1\%)} = 2 + \int_0^{-1} e^x\,dx = 2 + [e^x]_{x=0}^{x=-1} = 1 + e^{-1},$$

we conclude that $f(1,2) = 1 + e^{-1}$. (1%) □

Remark: The function $f(x, y)$ can be explicitly determined. Indeed, for all $x_0 \in \mathbb{R}$, we have

$$f(x_0, 0) = f(0,0) + \int_0^{x_0} f_x(x,0)\,dx = 2 + \int_0^{x_0} e^x\,dx = 1 + e^{x_0};$$

combining this formula with (b), we get $f(x,y) = f(x-y, 0) = 1 + e^{x-y}$ for all $(x,y) \in \mathbb{R}^2$.

4. (10%) Let $f(x,y) = x^2y + x^2 + y^2 - 2y$ for all $(x,y) \in \mathbb{R}^2$. Find all critical points of $f(x,y)$ and classify them as local maxima, local minima or saddle points.

Solution:

We have $f_x(x,y) = 2xy + 2x$ and $f_y(x,y) = x^2 + 2y - 2$. Solving $f_x(x,y) = f_y(x,y) = 0$, we get $(x,y) = (0,1)$, $(2,-1)$ or $(-2,-1)$; thus the critical points of f are $(0,1)$, $(2,-1)$ and $(-2,-1)$. We next use the second derivatives test to classify these critical points. We have $f_{xx}(x,y) = 2y+2$, $f_{xy}(x,y) = 2x$ and $f_{yy}(x,y) = 2$, and we set $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2$. Now:

- At $(0,1)$: Since $f_{xx}(0,1) = 4 > 0$ (or $f_{yy}(0,1) = 2 > 0$) and $D(0,1) = 4 \cdot 2 - 0^2 = 8 > 0$, we see that $(0,1)$ is a local minimum of f.
- At $(\pm 2,-1)$: Since $D(\pm 2,-1) = 0 \cdot 2 - (\pm 4)^2 = -16 < 0$, we see that $(\pm 2,-1)$ are saddle points of f. $\square$

Grading:

- 2% for correct $f_x(x,y)$ and $f_y(x,y)$ (1% each).
- 2% for finding the correct critical points of f (1% for $(0,1)$; 0.5% each for $(\pm 2,-1)$).
- 2% for correct $f_{xx}(x,y)$, $f_{xy}(x,y)$ and $f_{yy}(x,y)$ (1% for any first correct answer; 0.5% each for the rest two).
- 2% for any reasonable argument (such as the second derivatives test or the Taylor expansion) to classify the critical points already found, even if the critical points were not correctly identified (1% for any reasonable argument for local minima or local maxima; 1% for any reasonable argument for saddle points).
- 2% for the correct results of the classification of *correct* critical points of f (1% for $(0,1)$; 0.5% each for $(\pm 2,-1)$).
- If one finds k extra incorrect critical points with $k \geq 1$, then still use the above grading criteria to award points, except that the maximum total score he or she can get in this question will be $(10 - 0.5k)\%$ if $1 \leq k \leq 3$ and will be 8% if $k \geq 4$.

5. Suppose that all Mr.Munchlax does in a day is to play, to work and to sleep. To be more precise, on each day Mr.Munchlax spends

- x_1 hours on playing;
- x_2 hours on working;
- the remaining hours, $24 - x_1 - x_2$, on sleeping.

Suppose that sleeping is free, playing costs 100 dollars an hour and the hourly wage of working is 200 dollars. Furthermore, Mr.Munchlax spends all the money earned from working on playing.

The utility that Mr.Munchlax gets from sleeping and playing is given by a Cobb-Douglas-like function :

$$F(x_1, x_2) = x_1^{2/3} \cdot (24 - x_1 - x_2)^{1/3}.$$

(a) (2%) Write down the budget constraint as an equation of x_1 and x_2.

(b) (10%) By using the method of Lagrange multipliers, find the number of hours $x_2^\star$ that Mr.Munchlax has to work per day in order to maximize his utility under the budget constraint.

Solution:

(a) [(2M)] $100x_1 - 200x_2 = 0$, or simply $x_1 - 2x_2 = 0$.

Grading Scheme.

- (2M) All or nothing.

(b) Let $g(x_1, x_2) = x_1 - 2x_2$. Set $\nabla f = \lambda \nabla g$. We obtain

$$\boxed{\text{(2M)}}\begin{cases} \underbrace{\frac{2}{3}x_1^{-1/3}(24 - x_1 - x_2)^{1/3} - x_1^{2/3} \cdot \frac{1}{3}(24 - x_1 - x_2)^{-2/3}}_{\boxed{\text{(2M)}}} = \lambda(1) \\ \underbrace{-\frac{1}{3}x_1^{2/3} \cdot (24 - x_1 - x_2)^{-2/3}}_{\boxed{\text{(2M)}}} = \lambda(-2) \\ x_1 - 2x_2 = 0 \end{cases}$$

From the second equation, it is clear that $\lambda \neq 0$ in order to maximize the utility.
[(2M)] So we may (as we can) take the quotient of the first two equations to obtain

$$-2x_1^{-1}(24 - x_1 - x_2) + 1 = -\frac{1}{2} \Rightarrow 24 - x_1 - x_2 = \frac{3}{4}x_1$$

Now plug in $x_1 = 2x_2$, we have $24 - 3x_2 = \frac{3}{2}x_2$ and hence [(2M)] $x_2 = \frac{48}{9}$.

Since this is the only solution, this gives the number of working hours in order for Mr.Munchlax to maximize his utility in a day.

Grading Scheme.

- (2M) Correct f_{x_1}
- (2M) Correct f_{x_2}
- (2M) Correct system of equations, [-1M for each error]
- (2M) Take the quotient of first two (correctly computed) equations and take it correctly (or any reasonable and correct attempt)
- (2M) Correct answer

If a student obtained an incorrect constraint in (a), he/she can earn at most (but not necessarily) 5M from this part; coming from possibly the first three items listed above. Thank you for your cooperations.

6. Let $g(x, y) = xy$ be a function defined on a triangular region R with vertices $(3, 0), (0, 1)$ and $(0, 0)$. Let c be the average value of $g(x, y)$ over R, i.e.

$$c = \frac{\iint_R g(x, y)\, dA}{\iint_R 1\, dA}.$$

(a) (8%) Find the value of c.

(b) (4%) On the level curve $g(x, y) = c$, (x_0, y_0) is the point closest to the origin. It is known that (x_0, y_0) is in the interior of R. Show that $x_0 = y_0$ by the method of Lagrange multipliers.

Solution:

(a)

$$Area = \iint_R dA = \int_0^3 \int_0^{1-x/3} dydx = \frac{3}{2}. \quad (2\%)$$

It is OK that one answers it without integration.

$$\iint_R g(x, y) dA = \int_0^3 \int_0^{1-x/3} xydydx \quad (3\%)$$

$$= \int_0^3 \frac{x}{2}(1 - x/3)^2 dx$$

$$= \int_0^3 \frac{1}{18}(x^3 - 6x^2 + 9x)dx = \frac{1}{18}(\frac{x^4}{4} - 2x^3 + \frac{9}{2}x^2)|_0^3 = \frac{3}{8} \quad (1\%).$$

As $\iint_R dA = \frac{3}{2}$, one has average value is $\frac{1}{4}$ (2 %).

(b) Consider the function $f(x, y) = x^2 + y^2$ on the level curve $g(x, y) = c$. Method of Lagrange Multiplier shows that $2x = \lambda y$ and $2y = \lambda x$. (3 %)

It follows that $4x = \lambda^2 x$. The only reasonable point is obtained by $\lambda = 2$ and $x = y$. (1 %).

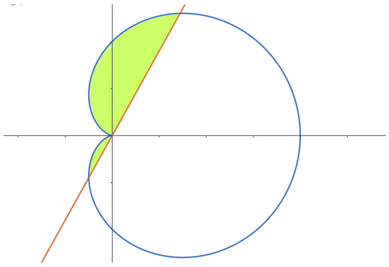
7. (a) (7%) Compute the integral $\int_0^1 \int_{x^{\frac{1}{3}}}^1 \frac{1}{y(1+y^2)}\, dydx$ by changing the order of integration.

Solution:

$$\int_0^1 \int_{x^{\frac{1}{3}}}^1 \frac{1}{y(1+y^2)} dydx = \int_0^1 \int_0^{y^3} \frac{1}{y(1+y^2)} dxdy \quad (4\%)$$

$$= \int_0^1 \frac{y^2}{1+y^2} dy = 1 - \int_0^1 \frac{1}{1+y^2} dy = 1 - \tan^{-1}(y)|_0^1 = 1 - \frac{\pi}{4} \quad (3\%)$$

(b) (9%) Find the area of the region that is inside the cardioid with polar equation $r = 1 + \cos\theta$ and on the left of the line $y = x$ on the xy-plane (See figure).



Solution:

$$Area = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_0^{1+\cos\theta} r dr d\theta \quad (4\%)$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}(1+\cos\theta)^2 d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2} + \cos\theta + \frac{1}{4}(1+\cos 2\theta) d\theta \quad (3\%)$$

$$= (\frac{3}{4} + \sin\theta + \frac{1}{8}\sin 2\theta)|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \frac{3}{4} - \sqrt{2}. \quad (2\%)$$

8. An economic experiment conducted at Taiwan Social Sciences Experimental Laboratory (TASSEL) randomly assigns each subject two positive numbers X and Y which represent subject reaction times for participating in one run of task A and B respectively. It is known that X and Y are independent with probability density functions

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}e^{-\frac{x}{2}} & \text{if } x \geq 0 \end{cases}, \quad f_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{3}e^{-\frac{y}{3}} & \text{if } y \geq 0 \end{cases} \quad \text{respectively.}$$

The experiment consists of 5 runs of task A and 3 runs of task B. Thus $T = 5X + 3Y$ represents the total time of the entire session. For any $t > 0$, the probability $T \leq t$ is $P(T \leq t) = \iint_D f_X(x)f_Y(y)\,dA$ where D is the region in the first quadrant of the xy−plane satisfying $5x + 3y \leq t$.

(a) (6%) Find the probability that the entire session does not exceed 15 minutes, $P(T \leq 15)$, and draw the corresponding integral region D.

(b) The random variable T has a probability density function $f_T(t)$ which is given by $f_T(t) = \frac{d}{dt}P(T \leq t)$ for $t > 0$ and $f_T(t) = 0$ for $t \leq 0$.

(i) (6%) By the change of variables $u = 5x+3y$, $v = x$, write $P(T \leq t) = \iint_D f_X(x)f_Y(y)\,dA$ as $\int_a^t \int_b^c g(u,v)\,dvdu$ for some function $g(u,v)$ and a, b, c.

(ii) (3%) Compute $\frac{d}{dt}P(T \leq t)$ for $t > 0$.

Solution:

(a) $P(T \leq 15) = \iint_D f_X(x)f_Y(y)\,dA$ where D is the region shown in the figure. (1 pt)

Solution 1:

Because $D = \{(x,y) \mid 0 \leq x \leq 3,\ 0 \leq y \leq 5 - \frac{5}{3}x\}$,

$$\iint_D f_X(x)f_Y(y)\,dA = \int_0^3 \int_0^{5-\frac{5}{3}x} \frac{1}{6}e^{-\frac{x}{2}}e^{-\frac{y}{3}}\,dydx \qquad \text{(2 pts)}$$

$$= \int_0^3 -\frac{1}{2}(e^{-\frac{5}{3}+\frac{1}{18}x} - e^{-\frac{1}{2}x})\,dx = \left(-9e^{-\frac{5}{3}}e^{\frac{1}{18}x} - e^{-\frac{1}{2}x}\right)\Big|_{x=0}^{x=3} \qquad \text{(2 pts)}$$

$$= 9e^{-\frac{5}{3}} - 10e^{-\frac{3}{2}} + 1. \qquad \text{(1 pt)}$$

Solution 2:

Because $D = \{(x,y) \mid 0 \leq y \leq 5,\ 0 \leq x \leq 3 - \frac{3}{5}y\}$,

$$\iint_D f_X(x)f_Y(y)\,dA = \int_0^5 \int_0^{3-\frac{3}{5}y} \frac{1}{6}e^{-\frac{x}{2}}e^{-\frac{y}{3}}\,dxdy \qquad \text{(2 pts)}$$

$$= \int_0^5 -\frac{1}{3}(e^{-\frac{3}{2}-\frac{1}{30}y} - e^{-\frac{1}{3}y})\,dy = \left(10e^{-\frac{3}{2}}e^{-\frac{1}{30}y} - e^{-\frac{1}{3}y}\right)\Big|_{y=0}^{y=5} \qquad \text{(2 pts)}$$

$$= 9e^{-\frac{5}{3}} - 10e^{-\frac{3}{2}} + 1. \qquad \text{(1 pt)}$$

(b) (i) $x = v, y = \frac{u-5v}{3}$ and $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3}$. (1 pt)

The corresponding region of D in the uv-plane is a triangle bounded by lines $v = 0$, $u = t$, and $u = 5v$. Hence

$$P(T \leq t) = \iint_D \frac{1}{6}e^{-\frac{x}{2}}e^{-\frac{y}{3}}\,dA = \int_0^t \int_0^{\frac{u}{5}} \frac{1}{6}e^{-\frac{v}{2}}e^{\frac{5v-u}{9}}\frac{1}{3}\,dvdu.$$

Therefore, $g(u,v) = \frac{1}{18}e^{-\frac{v}{2}}e^{\frac{5v-u}{9}} = \frac{1}{18}e^{-\frac{u}{9}+\frac{v}{18}}$.

(2 pts. 1 pt for plugging $x = v, y = \frac{u-5v}{3}$ into the function $f_X(x)f_Y(y)$, 1 pt for multiplying the Jacobian.)

And $a = b = 0$, $c = \frac{u}{5}$. (1 pt for each of a, b, c.)

(ii) For $t > 0$,

$$P(T \leq t) = \int_0^t e^{-\frac{u}{9}+\frac{u}{90}} - e^{-\frac{u}{9}}\, du = \int_0^t e^{-\frac{1}{10}u} - e^{-\frac{1}{9}u}\, du$$ (2 pts for doing the inner integration)

Hence $\frac{d}{dt}P(T \leq t) = e^{-\frac{t}{10}} - e^{-\frac{t}{9}}$ for $t > 0$. (1 pt for applying the FTC.)

(The answer $\frac{d}{dt}P(T \leq t) = \frac{1}{18}e^{-\frac{t}{9}} \int_0^{\frac{t}{5}} e^{\frac{v}{18}}\, dv$ gets 2 pts.)