112 模組05-09班 微積分3 期考解答和評分標準

1. Evaluate the limit or show that it doesn't exist.

   (a) (5%) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^4}$

   (b) (5%) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^4}$

Solution:

(a) Let $x = 0$. We have

$$\lim_{y\to 0} \frac{y^3}{y^4}$$

is divergent. Let $y = 0$. We have the limit converges to zero. Therefore, the limit doesn't exist.

(b) We have

$$\left|\frac{x^3}{x^2+y^4}\right| \le \frac{|x^3|}{x^2} = |x|$$

Thus, by squeeze theorem, the limit is zero.

(a)

- (2 pts) Claim the limit is divergent.
- (1 pt) Any sign of considering limit along paths.
- (2 pts) Successfully evaluate the limit along a path (one point for each limit)
- (-0.5 pt) Any minor computation.

(b)

- (2 pts) Claim the limit is convergent.
- (1 pt) Any sign of using inequality.
- (2 pts) Find the upper bound and the lower bound for the function $\dfrac{x^3}{x^2+y^4}$. One point for each bound.
- (-0.5 pt) Any minor computation.

2. Let surface S be given by $\sin(xyz) = x + 2y + 3z$. Suppose that near the point $(2,-1,0)$, the surface can be described by $z = z(x,y)$ as well as $y = y(x,z)$.

   (a) (6%) Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial y}{\partial x}$ at $(2,-1,0)$.

   (b) (7%) Suppose, when restricted to the surface S, a differentiable function $f(x,y,z)$ attains a local maximum at the point $(2,-1,0)$ with $f(2,-1,0) = 8$ and $f_x(2,-1,0) = 2$. Find $\nabla f(2,-1,0)$ and estimate $f(1.99,-0.99,0.01)$ by linear approximation.

Solution:

(a) We treat z as function of x. Then, the implicit differentiation with respect to x is

$$\cos(xyz)(yz + xy\frac{\partial z}{\partial x}) = 1 + 3\frac{\partial z}{\partial x}.$$

Plugin $(x,y,z) = (2,-1,0)$ and solve for $\frac{\partial z}{\partial x}$. We get $\frac{\partial z}{\partial x} = -1/5$.
Similarly, we treat y as function of x. We have the implicit differentiation with respect to x is

$$\cos(xyz)(yz + xz\frac{\partial y}{\partial x}) = 1 + 2\frac{\partial y}{\partial x}.$$

Plugin $(x,y,z) = (2,-1,0)$ and solve for $\frac{\partial y}{\partial x}$. We get $\frac{\partial y}{\partial x} = -1/2$.

(1.5 pt) Correctly carry out the implicit differentiation. (-0.5 pt for any computation error)

(0.5 pt) Use his/her answer. Correctly evaluate the equation at $(2,-1,0)$.

(1 pt) Use his/her answer. Correctly solve for $\partial z/\partial x$.

(a) Use the same grad scheme for $\partial y/\partial x$.

(b) Let $g(x) = \sin(xyz) - x - 2y - 3z$ and λ be the Lagrange multiplier of finding the extreme value of f subject to $g = 0$, i.e. $\nabla f = \lambda \nabla g$. In particular, we have

$$(f_x, f_y, f_z) = \lambda(g_x, g_y, g_z)$$

where

$$\begin{aligned} g_x &= \cos(xyz)(yz) - 1 \\ g_y &= \cos(xyz)(xz) - 2 \\ g_z &= \cos(xyz)(xy) - 3 \end{aligned}$$

Because the question asserts that $f_x(2,-1,0) = 2$, we can solve for λ by $2 = \lambda g_x(2,-1,0) = -\lambda$. That is $\lambda = -2$. Hence, $\nabla f = \lambda \nabla g = (2,4,10)$.
The linear approximation $L(x,y,z)$ of $f(x,y,z)$ at $(2,-1,0)$ is

$$L(x) = f(2,-1,0) + \nabla f \cdot (x-2, y+1, z).$$

Hence, $f(1.99,-0.99,0.01) \approx 8 + (2,4,10)\cdot(-0.01,0.01,0.01) = 8 + 0.12 = 8.12$.

(0.5 pt) Set g correctly.

(1 pt) Any sign of setting $\nabla f = \lambda \nabla g$.

(2 pt) Correctly compute ∇g. Computation error -0.5 pt.

(1 pt) Using his/her ∇g, he/she correctly solve λ.

(1 pt) Using his/her ∇g, he/she correctly solve ∇f.

(1 pt) Using his/her ∇f, correctly set up $L(x,y,z)$

(0.5 pt) Correctly evaluate $L(1.99,-0.99,0.01)$ using his/her $L(x,y,z)$.

3. Suppose that $F(x, y)$ has continuous second partial derivatives and $F(0,2) = 5$, $F_x(0,2) = -3$, $F_y(0,2) = -2$, $F_{xx}(0,2) = -1$, $F_{xy}(0,2) = 3$, $F_{yy}(0,2) = -2$.

(a) (4%) If $z = F(x, y)$, $x = 3\cos t$, $y = 2\sin t$, use the chain rule to find $\left.\dfrac{dz}{dt}\right|_{t=\pi/2}$.

(b) (7%) Use the chain rule to find $\left.\dfrac{d^2z}{dt^2}\right|_{t=\pi/2}$.

(c) (7%) Let $G(x, y) = F(x, y) - 2xy + 7x + 2y$. Show that $(0,2)$ is a critical point of $G(x, y)$. Use the second derivatives test to determine whether $(0,2)$ is a local maximum, local minimum, or saddle point.

Solution:

(a).

Using chain rule, $\dfrac{dz}{dt}$ is equal to

$$\frac{d}{dt}F(x(t), y(t)) = F_x(x(t), y(t)) \cdot x'(t) + F_y(x(t), y(t)) \cdot y'(t)$$

When $t = \dfrac{\pi}{2}$, $x = 0$ and $y = 2$. Also $x'(\pi/2) = -3$ and $y'(\pi/2) = 0$. Therefore

$$\left.\frac{dz}{dt}\right|_{t=\pi/2} = (-3) \cdot (-3) + (-2) \cdot (0) = 9.$$

□

(b).

To find $\dfrac{d^2z}{dt^2}$, we first note that by product rule

$$\frac{d}{dt}\left(\frac{dz}{dt}\right) = \left[\frac{d}{dt}F_x(x(t), y(t))\right] \cdot x'(t) + F_x(x(t), y(t)) \cdot x''(t) + \left[\frac{d}{dt}F_y(x(t), y(t))\right] \cdot y'(t) + F_y(x(t), y(t)) \cdot y''(t)$$

Then we use chain rule for the two derivatives

$$\frac{d}{dt}F_x(x(t), y(t)) = F_{xx}(x(t), y(t)) \cdot x'(t) + F_{xy}(x(t), y(t)) \cdot y'(t)$$

$$\frac{d}{dt}F_y(x(t), y(t)) = F_{yx}(x(t), y(t)) \cdot x'(t) + F_{yy}(x(t), y(t)) \cdot y'(t)$$

Recall that when $t = \dfrac{\pi}{2}$, $x = 0$ and $y = 2$. Also $x'(\pi/2) = -3$, $y'(\pi/2) = 0$, $x''(\pi/2) = 0$, and $y''(\pi/2) = -2$. Therefore

$$\left.\frac{d^2z}{dt^2}\right|_{t=\pi/2} = 9F_{xx}(0,2) - 2F_y(0,2) = -5$$

□

(c).

First we should compute $G_x(0,2)$ and $G_y(0,2)$.

$$G_x = F_x - 2y + 7 \quad , \quad G_y = F_y - 2x + 2$$

Hence at the point $(0,2)$, we see $G_x(0,2) = -3 - 4 + 7 = 0$ and $G_y(0,2) = -2 - 0 + 2 = 0$. Therefore $(0,2)$ is indeed a critical point of $G(x, y)$.

Next we need $G_{xx}(0,2)$, $G_{xy}(0,2)$, and $G_{yy}(0,2)$

$$G_{xx} = F_{xx} \quad , \quad G_{xy} = F_{xy} - 2 \quad , \quad G_{yy} = F_{yy}$$

Hence

$$G_{xx}(0,2) = -1 \quad , \quad G_{xy}(0,2) = 1 \quad , \quad G_{yy}(0,2) = -2$$

Using the second derivatives test

$$G_{xx}(0,2) < 0 \quad , \quad G_{xx}(0,2) \cdot G_{yy}(0,2) - [G_{xy}(0,2)]^2 = 1 > 0$$

The point $(0, 2)$ is a local maximum of the function $G(x, y)$. □

Grading:

• In 3(a), as long as chain rule is correct, -1 point for each mistake (so the checkboxes would be (1) all correct (2) 1 mistake (3) 2 mistakes (4) 3 mistakes). If $t = \pi/2$ is not plugged in, then we treat that as 3 mistakes.

• In 3(b), as long as chain rule is correct, 3 points for correctly expressing product rule. The rest of the problem is -1 point per mistake. If product rule is missing or incorrect, then student can get at most 3 points.

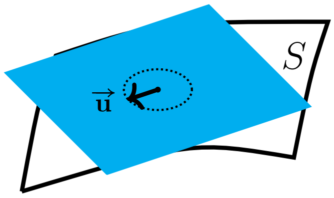
• If chain rule is incorrect, then student can get at most 1 point in (a) and 1 point in (b). Those points are from finding the correct x', y', x'', y'' values.

• In 3(c), 2 points for explaining the critical point and 5 points for using the second derivatives test correctly. If second derivatives test is incorrect, student can get at most 4 points (from critical point and correct second partials derivatives). Otherwise -1 point per error like before.

4. Consider a surface $S : x^3z - yz^3 - xy^3 + 1 = 0$ and a function $f(x,y,z) = \dfrac{z}{xy^4}$. You are traversing the surface S and seek the maximum rate of change of $f(x,y,z)$.

(a) (5%) Find an equation of the tangent plane to S at $(x,y,z) = (1,1,1)$.

(b) When you start at the point $(1,1,1)$ and move along the surface S, your unit tangent vector is denoted by $\mathbf{u} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Our goal is to find the maximum value of $D_{\mathbf{u}}f(1,1,1)$.



i. (5%) Write $D_{\mathbf{u}}f(1,1,1)$ as a function of a,b,c which is denoted by $F(a,b,c)$.

ii. (3%) Because $\mathbf{u} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ lies on the tangent plane to S at $(1,1,1)$ and $|\mathbf{u}| = 1$, list two constraints for a,b,c.

iii. (7%) Use the method of Lagrange multipliers to find the maximum value of $F(a,b,c)$ under constraints listed in part (ii).

Solution:

(a) Let $g(x,y,z) = x^3z - yz^3 - xy^3 + 1$. Then S is the level surface $g = 0$ and ∇g is normal to the level surface. Note that

$$\nabla g(x,y,z) = (3x^2z - y^3)\mathbf{i} - (z^3 + 3xy^2)\mathbf{j} + (x^3 - 3yz^2)\mathbf{k}, \qquad \text{(2 pts)}$$

and $\nabla g(1,1,1) = 2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$. (1 pt)

Thus the tangent plane of S at $(1,1,1)$ is

$$2(x-1) - 4(y-1) - 2(z-1) = 0 \implies x - 2y - z + 2 = 0. \qquad \text{(2 pts)}$$

(b) i

$$\nabla f(x,y,z) = -\frac{z}{x^2y^4}\,\mathbf{i} - \frac{4z}{xy^5}\,\mathbf{j} + \frac{1}{xy^4}\,\mathbf{k} \qquad \text{(2 pts)}, \qquad \nabla f(1,1,1) = -\mathbf{i} - 4\mathbf{j} + \mathbf{k}. \qquad \text{(1 pt)}$$

And $F(a,b,c) = D_{\mathbf{u}}f(1,1,1) = \nabla f(1,1,1)\cdot\mathbf{u} = -a - 4b + c$. (2 pts)

ii Because $\mathbf{u} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ is a unit vector lying on the tangent plane of S at $(x,y,z) = (1,1,1)$, constants a,b,c must satisfy the following equations

$$\nabla g(1,1,1)\cdot\mathbf{u} = 0 \implies G(a,b,c) = a - 2b - c = 0 \qquad \text{(2 pts)}$$

$$|\mathbf{u}| = 1 \implies H(a,b,c) = a^2 + b^2 + c^2 = 1 \qquad \text{(1 pt)}$$

iii By the method of Lagrange multipliers, to find the maximum value of $F(a,b,c)$ restricted to $G(a,b,c) = 0$ and $H(a,b,c) = 1$, we need to solve the system of equations:

$$\begin{cases} \nabla F(a,b,c) = \lambda\nabla G(a,b,c) + \mu\nabla H(a,b,c) \\ G(a,b,c) = a - 2b - c = 0 \\ H(a,b,c) = a^2 + b^2 + c^2 = 1 \end{cases} \qquad \text{(2 pts)}$$

$$\implies \begin{cases} -1 = \lambda + 2\mu a \\ -4 = -2\lambda + 2\mu b \\ 1 = -\lambda + 2\mu c \\ a - 2b - c = 0 \\ a^2 + b^2 + c^2 = 1 \end{cases} \qquad \text{(1 pt)}$$

Add the first and the third equations. Then we will have $\mu(a+c) = 0$ which implies $\mu = 0$ or $a + c = 0$. If $\mu = 0$, then the first equation solves $\lambda = -1$ but the second equation solves $\lambda = 2$ which is a contradiction. Thus $\mu \neq 0$ and $a + c = 0$.

Plug $c = -a$ into the fourth equation, we get $b = a$.

Then from the fifth equation we solve $(a,b,c) = \dfrac{1}{\sqrt{3}}(-1,-1,1)$, $\lambda = 1$, $\mu = \sqrt{3}$, or $(a,b,c) = \dfrac{1}{\sqrt{3}}(1,1,-1)$, $\lambda = 1$, $\mu = -\sqrt{3}$.

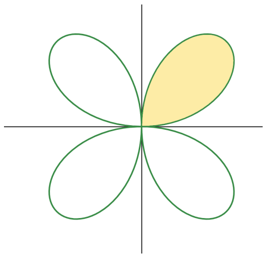
(1 pt for solving the system of equations.

1 pt for the answer $(a,b,c) = \dfrac{1}{\sqrt{3}}(-1,-1,1)$.

1 pt for the answer $(a,b,c) = \dfrac{1}{\sqrt{3}}(1,1,-1)$.)

Moreover, $F(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = 2\sqrt{3}$ and $F(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = -2\sqrt{3}$. Thus the maximum value of $D_{\mathbf{u}}f(1,1,1)$ is $2\sqrt{3}$. (1 pt)

5. Consider a plane region D bounded by the polar curve $r = \sin(2\theta)$ in the first quadrant.



(a) (6%) Find the area of D, $A(D)$.

(b) (6%) Find the average distance to the origin inside D which is $\dfrac{\iint_D \sqrt{x^2+y^2}\,dA}{A(D)}$.

Solution:

(a) The polar region D is $\{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le \sin(2\theta)\}$. Therefore,

$$A(D) = \iint_D 1\,dA \qquad \text{(1 pt)}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin(2\theta)} r\,drd\theta \qquad \text{(1 pt for ranges of } r \text{ and } \theta\text{. 1 pt for the Jacobian } r\text{)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin^2(2\theta)\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{4}\,d\theta = \left(\frac{\theta}{4} - \frac{\sin(4\theta)}{16}\right)\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}} \qquad \text{(2 pts)}$$

$$= \frac{\pi}{8} \qquad \text{(1 pt)}$$

(b)

$$\iint_D \sqrt{x^2+y^2}\,dA = \int_0^{\frac{\pi}{2}} \int_0^{\sin(2\theta)} r \cdot r\,drd\theta \qquad \text{(1 pt for ranges of } r \text{ and } \theta\text{. 1 pt for the Jacobian } r\text{)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}\sin^3(2\theta)\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{3}(1-\cos^2(2\theta))\sin(2\theta)\,d\theta \qquad \text{(1 pt for integrating } r^2\text{)}$$

Let $u = \cos(2\theta)$. Then $du = -2\sin(2\theta)\,d\theta$.

$$\int_0^{\frac{\pi}{2}} \frac{1}{3}(1-\cos^2(2\theta))\sin(2\theta)\,d\theta = \int_1^{-1} \frac{1}{3}(1-u^2)\frac{-1}{2}\,du = \frac{2}{9} \qquad \text{(2 pts for substitution)}$$

Thus the average distance over D is $\frac{\frac{2}{9}}{\frac{\pi}{8}} = \frac{16}{9\pi}$. (1 pt)

6. (a) (6%) Evaluate $\iiint_E 2\sqrt{z}\cos(x^2)\,dV$, where $E = \{(x,y,z) \mid 2y \le x \le 2,\ 0 \le y \le 1,\ 0 \le z \le 4\}$.

(b) (6%) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2} \cdot e^{(x^2+y^2+z^2)}\, dzdydx$.

Solution:

(a) E can be expressed by

$$E = \left\{(x,y,z) \,|\, 0 \le x \le 2,\ 0 \le y \le \frac{x}{2},\ 0 \le z \le 4\right\} \ (2\%).$$

So

$$\iiint_E 2\sqrt{z}\cos(x^2)\,dV = \int_0^4 \int_0^2 \int_0^{x/2} 2\sqrt{z}\cos(x^2)\,dy\,dx\,dz \ (1\%)$$

$$= \int_0^4 \int_0^2 \sqrt{z}\,x\cos(x^2)\,dx\,dz \ (1\%) = \left(\int_0^4 \sqrt{z}\,dz\right)\left(\int_0^2 x\cos(x^2)\,dx\right)$$

$$= \left(\frac{2}{3}z^{3/2}\Big]_0^4\right)\left(\frac{1}{2}\sin(x^2)\Big]_0^2\right) = \frac{8\sin 4}{3} \ (2\%).$$

(b)

$$E = \left\{(x,y,z) \,|\, 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2},\ \sqrt{x^2+y^2} \le z \le \sqrt{2-x^2+y^2}\right\}.$$

In spherical coordinate,

$$E = \left\{(\rho,\theta,\varphi) \,|\, 0 \le \rho \le \sqrt{2},\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le \frac{\pi}{4}\right\} \ (2\%).$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2+y^2}} \sqrt{x^2+y^2+z^2} \cdot e^{x^2+y^2+z^2}\, dzdydx$$

$$= \int_0^{\pi/2} \int_0^{\sqrt{2}} \int_0^{\pi/4} \rho^3 e^{\rho^2} \sin\varphi\,d\varphi\,d\rho\,d\theta \ (2\%)$$

$$= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{\pi/4} \sin\varphi\,d\varphi\right)\left(\int_0^{\sqrt{2}} \rho^3 e^{\rho^2}\,d\rho\right)$$

$$= \frac{\pi}{2}\left(-\cos\varphi\big]_0^{\pi/4}\right)\left(\frac{\rho^2 e^{\rho^2}}{2}\Big]_0^{\sqrt{2}} - \int_0^{\sqrt{2}} \rho e^{\rho^2}\,d\rho\right) \ (1\%)$$

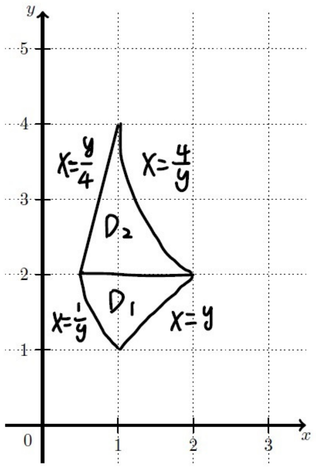
$$= \frac{\pi}{2}\left(1 - \frac{\sqrt{2}}{2}\right)\left(e^2 - \frac{e^{\rho^2}}{2}\Big]_0^{\sqrt{2}}\right) = \frac{\pi}{8}(2-\sqrt{2})(e^2+1). \ (1\%)$$

7. (a) (6%) Consider double integrals $\int_1^2 \int_{1/y}^{y} e^{xy}\,dx\,dy = \iint_{D_1} e^{xy}\,dA$ and $\int_2^4 \int_{y/4}^{4/y} e^{xy}\,dx\,dy = \iint_{D_2} e^{xy}\,dA$. Draw the regions D_1 and D_2.

(b) (9%) Evaluate $\int_1^2 \int_{1/y}^{y} e^{xy}\,dx\,dy + \int_2^4 \int_{y/4}^{4/y} e^{xy}\,dx\,dy$.

Solution:

(a)



Please draw the curves $x = 1/y$ for $1 \le y \le 2$ (1%) and $x = 4/y$ for $2 \le y \le 4$ (1%), the line segments $x = y$ for $1 \le y \le 2$ (1%) and $x = y/4$ for $2 \le y \le 4$ (1%) and point out the region D_1 (1%) and D_2 (1%).

(b) Set

$$u = xy, \quad v = \frac{y}{x} \text{ (1\%)}.$$

Then we have

$$D_1 \cup D_2 = \{(u, v) \mid 1 \le u \le 4,\ 1 \le v \le 4\}.\text{(2\%)}.$$

Since $x, y > 0$, we have $x = \sqrt{\frac{u}{v}}$ and $y = \sqrt{uv}$. The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & \frac{-1}{2}\frac{\sqrt{u}}{v\sqrt{v}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix} = \frac{1}{2v} \quad \text{(2\%)}$$

So

$$\begin{aligned} &\int_1^2 \int_{1/y}^{y} e^{xy}\,dx\,dy + \int_2^4 \int_{y/4}^{4/y} e^{xy}\,dx\,dy = \iint_{D_1 \cup D_2} e^{xy}\,dA \\ = &\int_1^4 \int_1^4 e^u \left|\frac{1}{2v}\right| du\,dv \text{ (2\%)} \\ = &\frac{1}{2}\left(\int_1^4 e^u\,du\right)\left(\int_1^4 \frac{1}{v}\,dv\right) = (e^4 - e^1) \cdot \ln 2. \text{ (2\%)} \end{aligned}$$