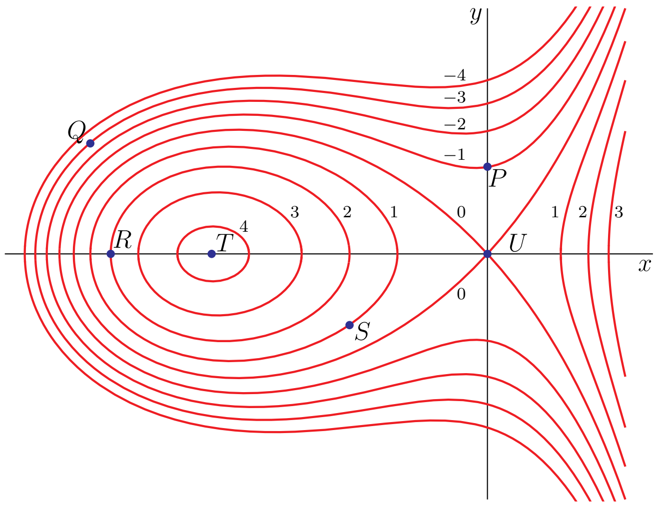
1. (a) (8%) Some level curves of a differentiable function $f(x, y)$ are plotted on the xy-plane below.



It is given that a local maximum of $f(x, y)$ occurs at point T.

For each of the following statements, circle all the alphabets for which the corresponding point(s) in the contour diagram satisfies the statement. **No credits would be given to partially correct/incorrect answer(s).**

(Sample) This point lies on the y-axis.	(P)	Q	R	S	T	(U)
(i) ∇f is the zero vector at this point.	P	Q	R	S	(T)	(U)
(ii) f has a saddle point at this point.	P	Q	R	S	T	(U)
(iii) The directional derivative of f in the direction $\langle 0, -1\rangle$ is negative at this point.	P	Q	R	(S)	T	U
(iv) The directional derivative of f in the direction $\langle 0, -1\rangle$ is zero at this point.	P	Q	(R)	S	(T)	(U)

Solution:

(i) ∇f is zero at critical points. The point T is a local maximum and the point U is a saddle point. The remaining points P, Q, R, S, are not critical points.

(iii) The directional derivative of f in the direction $\langle 0, -1\rangle$ is $\nabla f \cdot \langle 0, -1\rangle = -f_y$. It is negative if and only if $f_y > 0$.

We have $f_y(x, y) > 0$ if f increases as you move vertically upward through (x, y). Looking at the diagram, we see

$$f_y(P) < 0 \qquad f_y(Q) < 0 \qquad f_y(R) = 0 \qquad f_y(S) > 0 \qquad f_y(T) = 0 \qquad f_y(U) = 0$$

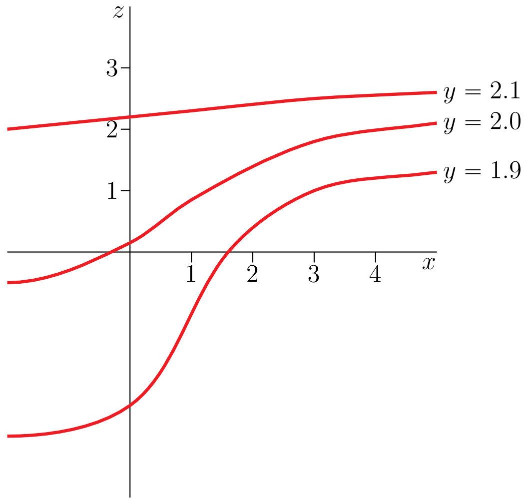
So only S works.

(iv) The directional derivative of f in the direction $\langle 0, -1\rangle$ is $\nabla f \cdot \langle 0, -1\rangle = -f_y$. It is zero if and only if $f_y = 0$.

From the analysis in (iii), the answer is R, T and U.

Marking Scheme. In part (a), each problem 2 points. No partial credits.

(b) (3%) The diagram below shows three "y-traces/slices" of a graph $z = F(x, y)$ plotted on the xz-plane. i.e. the intersection of the surface $z = F(x, y)$ with $y = k$ for $k = 1.9, 2, 2.1$ respectively. For each of the following statements below, circle the *best* answer.



(i) the first order partial derivative $F_x(1, 2)$ is positive/negative/zero (circle one)

(ii) F has a critical point at $(2, 2)$ true/false (circle one)

(iii) the second order partial derivative $F_{xy}(1, 2)$ is positive/negative/zero (circle one)

Solution:

(i) The function $z = F(x, 2)$ is increasing at $x = 1$, because the $y = 2.0$ graph in the diagram has positive slope at $x = 1$. So $F_x(1, 2) > 0$.

(ii) The function $z = F(x, 2)$ is also increasing (though slowly) at $x = 2$, because the $y = 2.0$ graph in the diagram has positive slope at $x = 2$. So $F_x(2, 2) > 0$. So F does *not* have a critical point at $(2, 2)$.

(iii) From the diagram the looks like $F_x(1, 1.9) > F_x(1, 2.0) > F_x(1, 2.1)$. That is, it looks like the slope of the $y = 1.9$ graph at $x = 1$ is larger than the slope of the $y = 2.0$ graph at $x = 1$, which in turn is larger than the slope of the $y = 2.1$ graph at $x = 1$. So it looks like $F_x(1, y)$ decreases as y increases through $y = 2$, and consequently $F_{xy}(1, 2) < 0$.

Marking Scheme. In part (b), each problem 1 point. No partial credits.

2. Let $f(x,y,z)$ be a function with continuous partial derivatives. Suppose that $f\left(0,0,\frac{1}{2}\right)=k$ and $f_z\left(0,0,\frac{1}{2}\right)\neq 0$.

(a) Let $\mathcal{C}$ be a space curve whose parametrization is given by

$$\mathbf{r}(\theta)=\langle(1-2\cos\theta)\cos\theta,(1-2\cos\theta)\sin\theta,\cos\theta\rangle.$$

It is known that $\mathcal{C}$ lies on the level surface $f(x,y,z)=k$.

(i) (2%) Find θ_1,θ_2 such that $0\le\theta_1<\theta_2\le 2\pi$ and $\mathbf{r}(\theta_1)=\mathbf{r}(\theta_2)=\left\langle 0,0,\frac{1}{2}\right\rangle$.

(ii) (8%) Find $\mathbf{r}'(\theta_1)$ and $\mathbf{r}'(\theta_2)$. Hence, find an equation of the tangent plane of $f(x,y,z)=k$ at $\left(0,0,\frac{1}{2}\right)$.

(b) (6%) Let g be a two-variable function with continuous partial derivatives such that $g(0,0)=0$ and $\nabla g(0,0)=\langle 1,2\rangle$. It is known that $f(x,y,z)$ attains a global maximum value at $\left(0,0,\frac{1}{2}\right)$ when it is restricted to the curve formed by the intersection of $z=\frac{1}{2}$ and $g(x^2+3y,x+Ay)=0$. Find the value of A.

Solution:

(a) (i) $\cos\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$ or $\frac{5\pi}{3}$. One can check that $\mathbf{r}(\frac{\pi}{3})=\mathbf{r}(\frac{5\pi}{3})=\langle 0,0,\frac{1}{2}\rangle$. Hence $\theta_1=\frac{\pi}{3}$ and $\theta_2=\frac{5\pi}{3}$.

Marking scheme:

- For this part, it is allowed to write down the answer without giving any argument.
- (1%) If the answer is partially correct, or in the wrong range.

(ii)

$$\begin{aligned}&\mathbf{r}'(\frac{\pi}{3})\\&=\left\langle\frac{d}{d\theta}[(1-2\cos\theta)\cos\theta],\frac{d}{d\theta}[(1-2\cos\theta)\sin\theta],\frac{d}{d\theta}[\cos\theta]\right\rangle\Big|_{\theta=\frac{\pi}{3}}\\&=\frac{\sqrt{3}}{2}\langle 1,\sqrt{3},-1\rangle,\end{aligned}$$

$$\begin{aligned}&\mathbf{r}'(\frac{5\pi}{3})\\&=\left\langle\frac{d}{d\theta}[(1-2\cos\theta)\cos\theta],\frac{d}{d\theta}[(1-2\cos\theta)\sin\theta],\frac{d}{d\theta}[\cos\theta]\right\rangle\Big|_{\theta=\frac{5\pi}{3}}\\&=\frac{\sqrt{3}}{2}\langle -1,\sqrt{3},1\rangle.\end{aligned}$$

The normal vector of the tangent plane is parallel to

$$\langle 1,\sqrt{3},-1\rangle\times\langle -1,\sqrt{3},1\rangle=2\sqrt{3}\langle 1,0,1\rangle.$$

Hence, the equation of the tangent plane is

$$x+(z-\frac{1}{2})=0.$$

Marking scheme:

- (2%) Correctly calculate $\mathbf{r}'(\theta_1)$ and $\mathbf{r}'(\theta_2)$. One point is granted if the answer is partially correct.
- (3%) A vector perpendicular to the tangent plane is found.
- (3%) Correctly write down the equation of the tangent plane.

(b) Since $\nabla f(0,0,\frac{1}{2})$ is perpendicular to the tangent plane of $f(x,y,z)=k$ at $(0,0,\frac{1}{2})$, we have

$$\nabla f(0,0,\frac{1}{2})=\ell\langle 1,0,1\rangle\text{ for some }\ell\in\mathbb{R}\setminus\{0\}.$$

Let $G(x, y, z) = g(x^2 + 3y, Ay + x)$. By the method of Lagrange multiplier, we must have $\nabla f(0, 0, \frac{1}{2}) = \lambda \nabla z(0, 0, \frac{1}{2}) + \mu \nabla G(0, 0, \frac{1}{2})$. That is,

$$\ell\langle 1, 0, 1\rangle = \lambda\langle 0, 0, 1\rangle + \mu\langle G_x(0, 0, \frac{1}{2}), G_y(0, 0, \frac{1}{2}), G_z(0, 0, \frac{1}{2})\rangle.$$

Hence,

$$0 = G_y(0, 0, \frac{1}{2}) \overset{\text{chain rule}}{=} 1 \cdot 3 + 2 \cdot A,$$

from which we have $A = -\frac{3}{2}$.

Marking scheme:

- (2%) For the geometric interpretation of $\nabla f(0, 0, \frac{1}{2})$. (That is , the student knows that it is parallel to $\langle 1, 0, 1\rangle$.)
- (2%) Correctly write down the relation among gradient vectors. It does not need to be identical to the above equation. Any equivalent statement is allowed.
- (2%) Correctly use the chain rule to find the value of A. One point is granted if the formula for the chain rule is correct.

3. Suppose that $f(x,y)$ is a differentiable function whose gradient vector equals

$$\nabla f(x,y) = (1 - 2x^2 - 2xy^2)e^{-x^2}\,\mathbf{i} + 2ye^{-x^2}\,\mathbf{j}.$$

(a) (6%)

(i) Use the method of Lagrange multipliers to find point(s) at which $f(x,y)$ possibly attains an extreme value when restricted to the line $y = 1$.

(ii) Use the method of Lagrange multipliers to find point(s) at which $f(x,y)$ possibly attains an extreme value when restricted to the line $y = -1$.

(b) (7%) Find all the critical points of $f(x,y)$ and classify each of them either as local minimum, local maximum, or saddle points.

(c) (5%) You are given that $f(x,y)$ attains an absolute minimum value at some point P in the region

$$D = \{(x,y) \in \mathbb{R}^2 \;:\; -1 \le y \le 1\}.$$

(i) Prove that P cannot be one of those points that you found in (a)(i) and (a)(ii). (Hint. Discuss the directions of ∇f at these points.)

(ii) Hence, find explicitly the point P.

Solution:

(a) (i) Let $g(x,y) = y$. By the method of Lagrange multipliers, to find extreme values of $f(x,y)$ restricted on $g(x,y) = 1$, we solve $\nabla f = \nabla g$ and $g(x,y) = 1$. (1 pt)

$$f_x = e^{-x^2}(1 - 2x^2 - 2xy^2) = \lambda g_x = 0$$
$$f_y = 2ye^{-x^2} = \lambda g_y = \lambda, \qquad y = 1. \qquad \text{(1 pt)}$$

Hence $(x,y) = (\frac{-1 \pm \sqrt{3}}{2}, 1)$. (1 pt)

(ii) Let $g(x,y) = y$. By the method of Lagrange multipliers, to find extreme values of $f(x,y)$ restricted on $g(x,y) = 1$, we solve $\nabla f = \nabla g$ and $g(x,y) = -1$. (1 pt)

$$f_x = e^{-x^2}(1 - 2x^2 - 2xy^2) = \lambda g_x = 0$$
$$f_y = 2ye^{-x^2} = \lambda g_y = \lambda, \qquad y = -1. \qquad \text{(1 pt)}$$

Hence $(x,y) = (\frac{-1 \pm \sqrt{3}}{2}, -1)$. (1 pt)

(b) To find critical points of f, we solve $\nabla f = \mathbf{0}$ which implies $y = 0$ and $x = \pm\frac{1}{\sqrt{2}}$. Hence critical points are $(x,y) = (\pm\frac{1}{\sqrt{2}}, 0)$. (1 pt)

$f_{xx} = e^{-x^2}(-6x - 2y^2 + 4x^3 + 4x^2y^2)$, (1 pt)

$f_{xy} = -4xye^{-x^2}$, $\quad f_{yy} = 2e^{-x^2}$. (1 pt)

$D(\frac{1}{\sqrt{2}}, 0) = -\frac{4\sqrt{2}}{e} < 0$. (1 pt) Hence $(x,y) = (\frac{1}{\sqrt{2}}, 0)$ is a saddle point. (1 pt)

$D(-\frac{1}{\sqrt{2}}, 0) = \frac{4\sqrt{2}}{e} > 0$ and $f_{xx}(-\frac{1}{\sqrt{2}}, 0) = 2\sqrt{\frac{2}{e}} > 0$. (1 pt)

Hence $f(-\frac{1}{\sqrt{2}}, 0)$ is a local minimum. (1 pt)

(c) (i) $\nabla f(\frac{-1 \pm \sqrt{3}}{2}, 1) = 2e^{-(\frac{-1 \pm \sqrt{3}}{2})^2}\,\mathbf{j}$ is in the positive direction of $\mathbf{j}$. Hence $D_{-\mathbf{j}}f(\frac{-1 \pm \sqrt{3}}{2}, 1) < 0$ which means that $f(x,y)$ decreases when (x,y) moves from $(\frac{-1 \pm \sqrt{3}}{2}, 1)$ toward the interior of D in the direction of $-\mathbf{j}$. Hence $f(\frac{-1 \pm \sqrt{3}}{2}, 1)$ can not be the absolute minimum on D. (2 pts)

Similarly, $\nabla f(\frac{-1\pm\sqrt{3}}{2}, 1) = -2e^{-(\frac{-1\pm\sqrt{3}}{2})^2}\mathbf{j}$ is in the negative direction of $\mathbf{j}$. Hence $D_{\mathbf{j}}f(\frac{-1\pm\sqrt{3}}{2}, 1) <$ 0 which means that $f(x,y)$ decreases when (x,y) moves from $(\frac{-1\pm\sqrt{3}}{2}, -1)$ toward the interior of D in the direction of $\mathbf{j}$. Hence $f(\frac{-1\pm\sqrt{3}}{2}, -1)$ can not be the absolute minimum on D. (2 pts)

(ii) The absolute minimum either occurs at some critical point or is the extreme value on the boundary of D. Because we have shown that the extreme values on the boundary of D can not be absolute minimum, the absolute minimum must occur at some critical point. Moreover, $(\frac{1}{\sqrt{2}}, 0)$ is a saddle point. Thus $f(\frac{1}{\sqrt{2}}, 0)$ can not be absolute minimum.

Therefore, the absolute minimum occurs at $(x,y) = (-\frac{1}{\sqrt{2}}, 0)$. (1 pt)

4. It is given that a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$|f(x,y) - (1 + 2x + 3y)| \le \ln(1 + x^2 + y^2) \text{ for all } (x,y) \in \mathbb{R}^2.$$

(a) (2%) Find $f(0,0)$.

(b) (5%) Use the definition of partial derivatives to find $f_x(0,0)$ and $f_y(0,0)$.

(c) (1%) Define what it means for $f : \mathbb{R}^2 \to \mathbb{R}$ to be differentiable at $(0,0)$.

(d) (5%) Hence, determine whether f is differentiable at $(0,0)$.

Solution:

(a) (1M) Put $x = y = 0$ in the given inequality yields

$$|f(0,0) - 1| \le 0.$$

(1M) This forces $f(0,0) = 1$.

Grading Scheme.

- (1M) for the correct method.
- (1M) for the correct answer.

(b) (1M) Our goal is to compute $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0) - 1}{h}$.

(1M) Take $x = h$ and $y = 0$ in the given inequality $|f(h,0) - 1 - 2h| \le \ln(1 + h^2)$.

(1M) Dividing both sides by $|h|$, we obtain $\left|\frac{f(h,0) - 1}{h} - 2\right| \le \left|\frac{\ln(1 + h^2)}{h}.\right|$.

(1M) By L'Hospital's rule, we have $\lim_{h\to 0} \frac{\ln(1 + h^2)}{h} = \lim_{h\to 0} \frac{2h}{1 + h^2} = 0$.

(0.5M) Hence, by Squeeze Theorem, we have $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - 1}{h} = 2$.

(0.5M) Likewise, one can show that $f_y(0,0) = 3$.

Grading Scheme.

- (1M) for the correct definition of $f_x(0,0)$ (or $f_y(0,0)$)
- (1+1+1M) for a correct argument by Squeeze Theorem which includes (1) Plugging in $x = h$ and $y = 0$; (2) Dividing both sides by $|h|$; (3) Use L'Hospital's rule to compute the limit of $\ln(1 + h^2)/h$ as $h \to 0$.
- (0.5M+0.5M) for the correct values of $f_x(0,0)$ and $f_y(0,0)$.

(c) (1M) $f(x,y)$ is differentiable at $(0,0)$ if $\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2 + y^2}} = 0$ where $L(x,y)$ is the linearization of $f(x,y)$ at $(0,0)$.

Grading Scheme.

- (1M) All or nothing. (We accept anequivalent formulation)

(d) (1M) By (a) and (b), the linerization of f at $(0,0)$ is given by $L(x,y) = 1 + 2x + 3y$.

(1M) By dividing the both sides of the given inequality by $\sqrt{x^2 + y^2}$, we obtain $\left|\frac{f(x,y) - L(x,y)}{\sqrt{x^2 + y^2}}\right| \le \frac{\ln(1 + x^2 + y^2)}{\sqrt{x^2 + y^2}}$.

(1M) Note that, by passing to polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{\ln(1 + x^2 + y^2)}{\sqrt{x^2 + y^2}} = \lim_{r\to 0^+} \frac{\ln(1 + r^2)}{r}.$$

(1M) As L'Hospital's rule implies $\lim_{r\to 0^+} \frac{\ln(1+r^2)}{r} = 0$,

(1M) using Squeeze Theorem, we deduce that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} = 0$$

so the given function is differentiable at $(0,0)$.

Grading Scheme.

- (1M) Write down the linearization.
- (1M) Obtaining an inequality for $\frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}}$
- (1+1M) Computing the limit of $\frac{\ln(1+x^2+y^2)}{\sqrt{x^2+y^2}}$ by (1) passing to polar coordintaes (2) use L'Hospital's rule
- (1M) Complete the argument by Squeeze Theorem

5. (a) (8%) Evaluate $\int_0^1 \int_{\sqrt{x}}^2 e^{y^3}\, \mathrm{d}y\, \mathrm{d}x + \int_1^2 \int_1^{y^2} e^{y^3}\, \mathrm{d}x\, \mathrm{d}y$.

(b) (8%) Find the volume of the solid under the graph $z = \dfrac{1}{1+\sqrt{x^2+y^2}}$ and above the region

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2, -\sqrt{4-x^2} \le y \le 0\} \text{ on the } xy\text{-plane.}$$

Solution:

(i) $\int_0^1 \int_{\sqrt{x}}^2 e^{y^3} dydx = \iint_{D_1} e^{y^3} dA$ and $\int_1^2 \int_1^{y^2} e^{y^3} dxdy = \iint_{D_2} e^{y^3} dA$, where

$$D_1 = \{(x,y) \mid 0 \le x \le 1, \sqrt{x} \le y \le 2\}$$
$$D_2 = \{(x,y) \mid 1 \le y \le 2, 1 \le x \le y^2\}.$$

D_1 intersects D_2 only on the boundaries, and

$$D = D_1 \cup D_2 = \{(x,y) \mid 0 \le y \le 2, 0 \le x \le y^2\}.$$

Hence

$$\int_0^1 \int_{\sqrt{x}}^2 e^{y^3} dydx + \int_1^2 \int_1^{y^2} e^{y^3} dxdy$$
$$= \iint_D e^{y^3} dA = \int_0^2 \int_0^{y^2} e^{y^3} dxdy = \int_0^2 y^2 e^{y^3} dy = \frac{1}{3} e^{y^3}\Big|_0^2 = \frac{1}{3}(e^8 - 1).$$

Marking scheme:

- (2%) Correctly interpret the iterated integrals as double integrals. The regions must be correct. (1%) For each region.
- (2%) Correctly use the additivity on domains.
- (2%) Choose the right order of integration and write down the iterated integral $\int_0^2 \int_0^{y^2} e^{y^3} dxdy$.
- (2%) Correctly carry out the integration. No partial credit. If you derive a wrong integral, you will get (0%) even if you do the integral correctly. (i.e. if you get a wrong answer, you will lose 2%.)
- (-2%) The process of deriving $\int_0^2 \int_0^{y^2} e^{y^3} dxdy$ accounts for 6% according to the criteria above. For deriving **correct** iterated integral without any explanation. (a simple figure of integration bounds is acceptable and you will **NOT** lose this 2%)
- (-5%) The process of deriving $\int_0^2 \int_0^{y^2} e^{y^3} dxdy$ accounts for 6% according to the criteria above. For deriving **incorrect** iterated integral without reasonable explanation, but the student understands (i) additivity on domains, or (ii) change from iterated integral into double integral

(ii) What we want to find is $\iint_D \dfrac{1}{1+\sqrt{x^2+y^2}} dA$. We have

$$D = \{(r,\theta) \mid \frac{3\pi}{2} \le \theta \le 2\pi, 0 \le r \le 2\}.$$

Changing to polar coordinates,

$$\iint_D \frac{1}{1+\sqrt{x^2+y^2}} dA \overset{r \geqq 0}{=} \int_{\frac{3\pi}{2}}^{2\pi} \int_0^2 \frac{r}{1+r} drd\theta = \int_{\frac{3\pi}{2}}^{2\pi} [r - \ln(1+r)]|_0^2\, d\theta = (2 - \ln 3)|_{\frac{3\pi}{2}}^{2\pi} = (1 - \frac{\ln 3}{2})\pi.$$

Marking scheme:

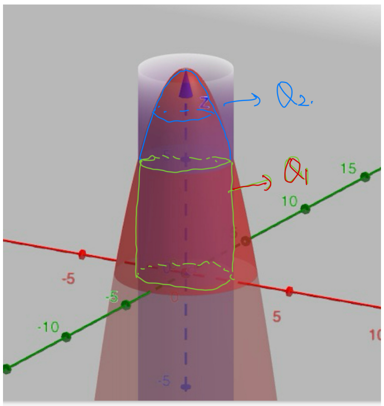
- (2%) Interpret the volume as the double integral.
- (2%) Correctly write down the region in polar coordinates.
- (2%) The change-to-polar-coordinates formula is correctly applied.
- (2%) Correctly carry out the integration. No partial credit. If you derive a wrong integral, you will get (0%) even if you do the integral correctly. (i.e. if you get a wrong answer, you will lose 2%.)
- Example : if you claim the region is $\theta \in [\pi, 2\pi]$, you will lose 2% for **correctly write down the region in polar coordinates** and 2% for **correctly carry out the integration**, so you will get 8%-2%-2%=4% for (b).

6. (a) (8%) Use cylindrical coordinates to find the volume of the solid Q that lies below $z = 9 - x^2 - y^2$, inside the cylinder $x^2 + y^2 = 4$ and above $z = 0$.

(b) (8%) Use spherical coordinates to find $\iiint_R \frac{1}{\sqrt{x^2+y^2+z^2}} \, dV$ where R is the solid region that lies inside the sphere $x^2 + y^2 + z^2 = 8$ and above the plane $z = 2$.

Solution:

(a)



The surface $z = 9 - x^2 - y^2$ and the cylinder $x^2 + y^2 = 4$ intersect at $z = 9 - 4 = 5$. From the graph, we can see that the region Q is

$Q = Q_1 \cup Q_2$ where

$Q_1 = \{(x,y,z)|0 \le x^2 + y^2 \le 4, 0 \le z \le 5\}$ and $Q_2 = \{(x,y,z)|5 \le z \le 9 - x^2 - y^2, x^2 + y^2 \le 4\}$

In cylindrical coordinates, it is

$Q_1 = \{(r,\theta,z)|0 \le r \le 2, 0 \le \theta \le 2\pi, 0 \le z \le 5\}$ and $Q_2 = \{(r,\theta,z)|5 \le z \le 9 - r^2, 0 \le r \le 2, 0 \le \theta \le 2\pi\}$.

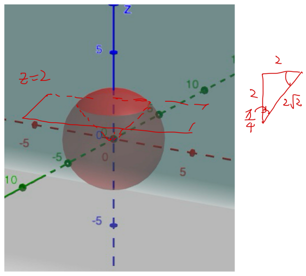
The volume of Q is given by the integral

$$\begin{aligned}
&\int\int\int_Q dV \\
&= \int\int\int_{Q_1} dV + \int\int\int_{Q_2} dV \\
&= \int_0^{2\pi}\int_0^2\int_0^5 r dz dr d\theta + \int_0^{2\pi}\int_0^2\int_5^{9-r^2} r dz dr d\theta \\
&= \int_0^{2\pi}\int_0^2 rz\Big|_{z=0}^{z=5} dr d\theta + \int_0^{2\pi}\int_0^2 zr\Big|_{z=5}^{z=9-r^2} dr d\theta \\
&= \int_0^{2\pi}\int_0^2 5r dr d\theta + \int_0^{2\pi}\int_0^2 (9-r^2)r - 5r dr d\theta \\
&= \int_0^{2\pi} \frac{5r^2}{2}\Big|_{r=0}^{r=2} d\theta + \int_0^{2\pi} 2r^2 - \frac{r^4}{4}\Big|_{r=0}^{r=2} d\theta \\
&= 20\pi + 8\pi \\
&= 28\pi
\end{aligned}$$

Marking Scheme.

- (2%) For finding the right region Q_1 in cylindrical coordinate.
- (2%) For finding the right region Q_2 in cylindrical coordinate..
- (2%) For finding the right triple integral on Q_1
- (2%) For finding the right triple integral on Q_2

(b)



Recall R is the solid region that lies inside the sphere $x^2 + y^2 + z^2 = 8$ and above the plane $z = 2$. Note that $x^2 + y^2 + z^2 = 8$ and $z = 2$ intersects at $x^2 + y^2 = 4$ We have $R = \{(x, y, z)|2 \le z \le \sqrt{8 - x^2 - y^2}, 0 \le x^2 + y^2 \le 4\}$

Recall the spherical coordinate, $x = \rho\sin(\phi)\cos(\theta), y = \rho\sin(\phi)\sin(\theta), z = \rho\cos(\phi)$. So $2 \le z \le \sqrt{8 - x^2 - y^2}$ can be expressed as $2 \le \rho\cos(\phi) \le \sqrt{8 - \rho^2\sin^2(\phi)}$. So $\frac{2}{\cos(\phi)} \le \rho \le 2\sqrt{2}$.

From $0 \le x^2 + y^2 \le 4$, we have $0 \le \theta \le 2\pi$ and $0 \le \rho\sin(\phi) \le 2$. Hence $0 \le \frac{2}{\cos(\phi)} \le \rho \le \frac{2}{\sin(\phi)}$, $0 \le \tan(\phi) \le 1$ and $0 \le \phi \le \frac{\pi}{4}$ In spherical coordinate, $R = \{(\rho, \theta, \phi)|\frac{2}{\cos(\phi)} \le \rho \le 2\sqrt{2}, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{4}\}$

We want to evaluate the integral:

$$\int\int\int_R \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV$$

$$\begin{aligned}
&\int\int\int_R \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV \\
=&\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{2}{\cos(\phi)}}^{2\sqrt{2}} \frac{1}{\rho}\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta \\
=&\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\rho^2}{2}\sin(\phi)\Big|_{\frac{2}{\cos(\phi)}}^{2\sqrt{2}} d\phi\,d\theta \\
=&\int_0^{2\pi}\int_0^{\frac{\pi}{4}} (4\sin(\phi) - \frac{2\sin(\phi)}{\cos^2(\phi)})\,d\phi\,d\theta \\
=&\int_0^{2\pi} (-4\cos(\phi) - \frac{2}{\cos(\phi)})\Big|_{\phi=0}^{\phi=\frac{\pi}{4}} d\theta \\
=&2\pi[(-4\frac{\sqrt{2}}{2} - \frac{2}{\frac{\sqrt{2}}{2}}) - (-4 - 2))] \\
=&-8\pi\sqrt{2} + 12\pi
\end{aligned}$$

Marking Scheme.

- (2%) For finding the right region R in spherical coordinate.
- (1%) For getting the right Jacobian factor $\rho^2\sin(\phi)$ in spherical coordinate.
- (2%) For finding the right inner integral $d\rho$.
- (2%) For finding the right integral $d\phi$
- (1%) For finding the right integral $d\theta$

7. Consider the following two changes of variables :

$$T(s,t) = (s+2t,\, 6s+2t); \quad \Phi(u,v) = (uv,\, uv^2).$$

(a) (2%) For any region D on the st-plane with nonzero area, find the ratio $\dfrac{\text{Area}(T(D))}{\text{Area}(D)}$.

(b) Let $\mathcal{R} = [1,3] \times [2,5]$ be a rectangle on uv-plane.

(i) (6%) Find $\text{Area}(\Phi(\mathcal{R})) = \iint_{\Phi(\mathcal{R})} 1\, dA$. i.e. the area of the image of $\mathcal{R}$ under Φ.

(ii) (2%) Hence or otherwise, find $\text{Area}(T(\Phi(\mathcal{R})))$. i.e. the area of the image of $\Phi(\mathcal{R})$ under T.

Solution:

(a) Since T is a linear transformation, it induces a constant area scale factor which equals the absolute value of the Jacobian $|J_T|$.

(1M) As $J_T = \begin{vmatrix} 1 & 2 \\ 6 & 2 \end{vmatrix} = -10$,

(1M) we deduce that $\dfrac{\text{Area}(T(D))}{\text{Area}(D)} = |J_T| = 10$.

Grading Scheme.

- (1M) Definition of Jacobian
- (1M) Correct answer

(b) (i) The transformation that underlies Φ is given by $x = uv$, $y = uv^2$.

(2M) Its Jacobian equals $J_\Phi = \begin{vmatrix} v & u \\ v^2 & 2uv \end{vmatrix} = uv^2$.

(3M) Therefore, $\iint_{\Phi(\mathcal{R})} 1\, dA = \int_2^5 \int_1^3 1 \cdot uv^2\, du\, dv$.

(1M) By a direct computation, we obtain $\int_2^5 \int_1^3 1 \cdot uv^2\, du\, dv = \dfrac{3^2 - 1^2}{2} \cdot \dfrac{5^3 - 2^3}{3} = 156$.

Grading Scheme.

- (2M) Correct Jacobian
- (1M+1M+1M) Correct transformation of the area integral (1) bounds for du, (2) bounds for dv, (3) no missing Jacobian
- (1M) Correct answer.

(ii) (1M) By (a), we have $\text{Area}(T(\Phi(\mathcal{R}))) = 10 \cdot \text{Area}(\Phi(\mathcal{R}))$.

(1M) By (b)(i), this equals $10 \cdot 156 = 1560$.

Grading Scheme.

- (1M) Using (a).
- (1M) Correct answer.