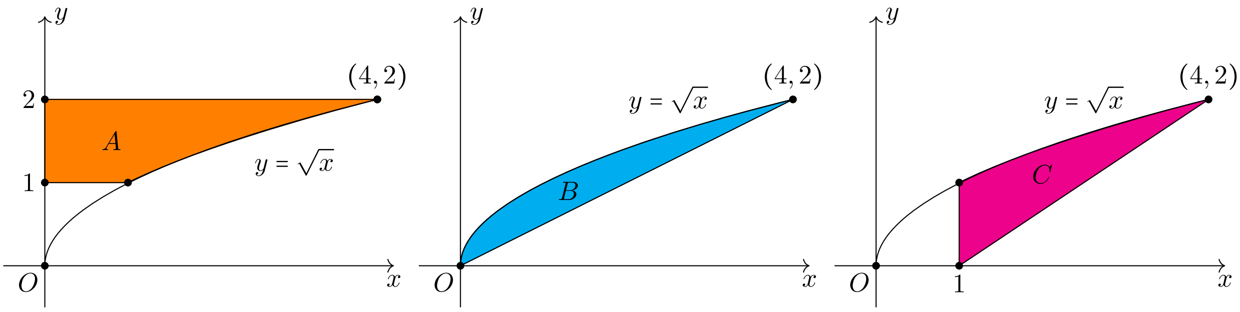
1. (10%) A, B, and C are the area of the regions in the pictures below. Evaluate the values of A, B, and C. Then list them from the biggest to the smallest.



Solution:

We compute A, B and C. To compute A, We first compute $\int_1^4 \sqrt{x}dx$.

$$\int_1^4 \sqrt{x}dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}.$$

Noting that the rectangle $[0,4] \times [0,2]$ consists of A, rectangle $[0,1] \times [0,1]$, and the region below the function $\sqrt{x}$ on $[1,4]$, we obtain that

$$A = 8 - 1 - \int_1^4 \sqrt{x}dx = 7 - \frac{14}{3} = \frac{7}{3}.$$

To compute B, we first compute $\int_0^4 \sqrt{x}dx$

$$\int_0^4 \sqrt{x}dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^4 = \frac{16}{3}.$$

Noting that the region below the function $\sqrt{x}$ on $[0,4]$ consists of B and the triangle with base 4 and height 2, we obtain

$$B = \int_0^4 \sqrt{x}dx - 4 = \frac{16}{3} - 4 = \frac{4}{3}.$$

Finally, we compute C. We first note $\int_1^4 \sqrt{x}dx = \frac{14}{3}$. Then we note that the region below the function $\sqrt{x}$ on $[1,4]$ consists of C and the triangle with base 3 and height 2, we obtain

$$C = \int_1^4 \sqrt{x}dx - 3 = \frac{14}{3} - 3 = \frac{5}{3}.$$

Comparing the values of A, B and C, we obtain $A > C > B$.

2. Let $F(x) = \int_{e \cdot x}^{e^x} \frac{\tan^{-1} t}{\ln(2+t)}\, dt$.

(a) (2%) Find the value of $F(1)$.

(b) (4%) Use the Fundamental Theorem of Calculus to find $F'(x)$.

(c) (3%) Find the value of $F'(0)$.

Solution:

(a)

$$F(1) = \int_e^e \frac{\tan^{-1}(t)}{\ln(2+t)} dt = 0$$

(b) We first observe

$$\begin{aligned} F(x) &= \int_{ex}^{e^x} \frac{\tan^{-1}(t)}{\ln(2+t)} dt \\ &= \int_0^{e^x} \frac{\tan^{-1}(t)}{\ln(2+t)} dt + \int_{ex}^{0} \frac{\tan^{-1}(t)}{\ln(2+t)} dt \\ &= \int_0^{e^x} \frac{\tan^{-1}(t)}{\ln(2+t)} dt - \int_0^{ex} \frac{\tan^{-1}(t)}{\ln(2+t)} dt. \end{aligned}$$

Therefore, by Fundamental Theorem of Calculus,

$$F'(x) = \frac{\tan^{-1}(e^x)}{\ln(2+e^x)} e^x - \frac{\tan^{-1}(ex)e}{\ln(2+ex)}$$

(c)

$$F'(0) = \frac{\tan^{-1}(1)}{\ln(3)} - \frac{\tan^{-1}(0)e}{\ln(2)} = \frac{\pi}{4\ln(3)}.$$

3. Evaluate the integrals.

(a) (9%) $\int 2x \tan^{-1} x \; dx$

(b) (9%) $\int_0^{\pi/4} \tan^3 \theta \; d\theta$

(c) (9%) $\int \frac{x^2}{(x+1)(x^2+1)} \; dx$

Solution:

(a) This integral can be evaluated with multiple methods. We provide two here.

Method 1: Integration by parts then long division.

$$\int 2x \tan^{-1} x \; dx = \int \tan^{-1} x \; d(x^2) = x^2 \tan^{-1} x - \int \frac{x^2}{x^2+1} \; dx$$
$$= x^2 \tan^{-1} x - \int \left(1 - \frac{1}{x^2+1}\right) \; dx = x^2 \tan^{-1} x - x + \tan^{-1} x + C$$

Method 2: Trigonometric substitution then integration by parts.

$$\int 2x \tan^{-1} x \; dx = \int 2(\tan\theta) \cdot \theta \cdot \sec^2\theta \; d\theta \text{ with } x = \tan\theta, \theta \in (-\pi/2, \pi/2)$$
$$= \int \theta d(\tan^2\theta) = \theta \tan^2\theta - \int \tan^2\theta \; d\theta$$
$$= \theta \tan^2\theta - \int (\sec^2\theta - 1) \; d\theta = \theta\tan^2\theta - \tan\theta + \theta + C$$
$$= x^2 \tan^{-1} x - x + \tan^{-1} x + C$$

□

Grading:

- Each correct integration technique applied will be (3%) for the student (as long as it is toward the correct answer, up to 6%). Use this grading method if the student did not reach a final answer or if they made too many minor mistakes.
- Each minor mistake (such as forgetting $+C$) is (-1%).
- If a student uses a different method, read their work to see if all the steps are correct. Each mistake is at most (-3%). Minor mistakes are (-1%) each.

(b) This integral can be evaluated with multiple methods. We provide three here.

Method 1: $\tan^2\theta = \sec^2\theta - 1$.

$$\int_0^{\pi/4} \tan^3\theta \; d\theta = \int_0^{\pi/4} (\tan\theta \sec^2\theta - \tan\theta) \; d\theta$$
$$= \int_0^{\pi/4} \tan\theta \sec^2\theta \; d\theta - \int_0^{\pi/4} \tan\theta \; d\theta$$
$$= \left[\frac{\tan^2\theta}{2}\right]_0^{\pi/4} - [\ln|\sec\theta|]_0^{\pi/4}$$
$$= \frac{1}{2} - \ln\sqrt{2} = \frac{1}{2} - \frac{1}{2}\ln 2$$

Method 2: Use $\sin\theta$ and $\cos\theta$.

$$\int_0^{\pi/4} \tan^3\theta \; d\theta = \int_0^{\pi/4} \frac{\sin^3\theta}{\cos^3\theta} \; d\theta$$
$$= \int_0^{\pi/4} \frac{1-\cos^2\theta}{\cos^3\theta} \; \sin\theta d\theta$$

$$= \int_1^{1/\sqrt{2}} -\frac{1-u^2}{u^3}\ du = \int_{1/\sqrt{2}}^1 (u^{-3} - u^{-1})\ du$$

$$= \left[\frac{u^{-2}}{-2} - \ln|u|\right]_{1/\sqrt{2}}^1$$

$$= -\frac{1}{2} + 1 + \ln(1/\sqrt{2}) = \frac{1}{2} - \ln\sqrt{2} = \frac{1}{2} - \frac{1}{2}\ln 2$$

Method 3: Use $\theta = \tan^{-1} x$.

$$\int_0^{\pi/4} \tan^3\theta\ d\theta = \int_0^1 \frac{x^3}{1+x^2}\ dx$$

$$= \int_0^1 \left(x - \frac{x}{1+x^2}\right)\ dx = \left[\frac{x^2}{2} - \frac{1}{2}\ln|1+x^2|\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{2}\ln 2$$

□

Grading:

- Each correct integration technique applied will be (3%) for the student (as long as it is toward the correct answer, up to 6%). Use this grading method if the student did not reach a final answer or if they made too many minor mistakes.
- Each minor mistake (such as forgetting $+C$) is (-1%).
- If a student uses a different method, read their work to see if all the steps are correct. Each mistake is at most (-3%). Minor mistakes are (-1%) each.

(c)

$$\frac{x^2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow A = 1/2,\ B = 1/2, \text{ and C=(-1)/2.}$$

Thus

$$\int \frac{x^2}{(x+1)(x^2+1)}dx = \frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{x}{x^2+1}dx - \int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + C$$

4. (9%) Fill in the blanks. Apply a suitable trigonometric substitution for each definite integral to find the value of the constants (Reminder: They can be negative). DO NOT evaluate the integrals.

(a) $\int_0^3 x^4\sqrt{9-x^2}\, dx = \int_0^{\pi/2} A\ \sin^B\theta\ \cos^C\theta\ d\theta.$

$A =$ ______ , $B =$ ______ , $C =$ ______

(b) $\int_0^{2/3} \frac{\sqrt{9x^2+4}}{x^3}\, dx = \int_0^{\pi/4} A\ \tan^B\theta\ \sec^C\theta\ d\theta.$

$A =$ ______ , $B =$ ______ , $C =$ ______

(c) $\int_{\sqrt{2}}^2 \frac{1}{(x^2-1)^3}\, dx = \int_{\pi/4}^{\pi/3} A\ \tan^B\theta\ \sec^C\theta\ d\theta.$

$A =$ ______ , $B =$ ______ , $C =$ ______

Solution:

(a)

$$\int_0^3 x^4\sqrt{9-x^2}dx = \int_0^{\pi/2} 81\sin^4\theta\cdot 3\cos\theta\cdot 3\cos\theta d\theta = \int_0^{\pi/2} 729\cdot\sin^4\theta\cdot\cos^2\theta d\theta.$$

Thus $A = 729$, $B = 4$, and $C = 2$.

(b)

$$\int_0^{\frac{2}{3}} \frac{\sqrt{9x^2+4}}{x^3}dx = \int_0^{\frac{\pi}{4}} \frac{2\sec\theta}{(8/27)\tan^3\theta}\cdot\frac{2}{3}\sec^2\theta d\theta = \int_0^{\pi/4} \frac{9}{2}\tan^{-3}\theta\sec^3\theta d\theta.$$

Thus $A = 9/2$, $B = -3$, $C = 3$.

(c)

$$\int_{\sqrt{2}}^2 \frac{1}{(x^2-1)^3}dx = \int_{\pi/4}^{\pi/3} \sec\theta\cdot\tan^{-5}\theta d\theta.$$

Thus $A = 1$, $B = -5$, $C = 1$.

5. Determine whether the improper integral is convergent or divergent with a limit. If it converges, evaluate the integral.

(a) (5%) $\int_0^1 \frac{2}{x \ln x}\, dx$

(b) (5%) $\int_0^\infty \frac{\tan^{-1} x}{x^2+1}\, dx$

Solution:

(a) We first compute $\int \frac{1}{x \ln x} dx$. We use substitution with $u = \ln(x)$. Then $du = \frac{1}{x} dx$, and we obtain

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{u} du = \ln(|u|) + C = \ln(|\ln(x)|) + C.$$

Next, we compute the given definite integral. Noting that the function $\frac{1}{x \ln x}$ is not defined at $x = 0$ and $x = 1$, the given integral is an improper integral. Since both lower and upper limits are improper, we split the integral into to parts

$$\int_0^1 \frac{1}{x \ln x} dx = \int_0^{\frac{1}{2}} \frac{1}{x \ln x} dx + \int_{\frac{1}{2}}^1 \frac{1}{x \ln x} dx.$$

Then we substitute the lower limit 0 with t with limit, and the upper limit 1 with s with limit.

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{1}{x \ln x} dx + \int_{\frac{1}{2}}^1 \frac{1}{x \ln x} dx &= \lim_{t \to 0^+} \int_t^{\frac{1}{2}} \frac{1}{x \ln x} dx + \lim_{s \to 1^-} \int_{\frac{1}{2}}^s \frac{1}{x \ln x} dx \\ &= \lim_{t \to 0^+} \ln(|\ln x|)\Big|_t^{\frac{1}{2}} + \lim_{s \to 1^-} \ln(|\ln(x)|)\Big|_{\frac{1}{2}}^s \\ &= \lim_{t \to 0^+} -\ln(|\ln(t)|) + \lim_{s \to 1^-} \ln(|\ln(s)|) \end{aligned}$$

Both limits on the last line diverges, and therefore the improper integral $\int_0^1 \frac{1}{x \ln x} dx$ is divergent.

(b) We first compute $\int \frac{\tan^{-1} x}{x^2+1} dx$. We use substitution with $u = \tan^{-1}(x)$. Then $du = \frac{1}{x^2+1} dx$, and we obtain

$$\int \frac{\tan^{-1} x}{x^2+1} dx = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} (\tan^{-1} x)^2 + C.$$

Next, we compute the given definite integral. We substitute the upper limit ∞ with t with limit.

$$\begin{aligned} \int_0^\infty \frac{\tan^{-1} x}{x^2+1} dx = \lim_{t \to \infty} \int_0^t \frac{\tan^{-1} x}{x^2+1} dx &= \lim_{t \to \infty} \frac{1}{2} (\tan^{-1} x)^2 \Big|_0^t \\ &= \lim_{t \to \infty} \frac{1}{2} (\tan^{-1} t)^2 = \frac{1}{2} \left(\frac{\pi}{2}\right)^2. \end{aligned}$$

6. Solve the initial value problems.

(a) (7%) Find the function $y(x)$ such that

$$xy' = y + x^2 \sin x \quad , \quad y(\pi) = 0.$$

(b) (8%) Find the function $f(x)$ such that

$$f(x) = 3 + \int_2^x \frac{dt}{(tf(t))^2}$$

(Hint: Differentiate to get a differential equation. You can find the value of $f(2)$.)

Solution:

(a)

$$xy' = y + x^2 \mathrm{sin} x \Rightarrow y' - \frac{1}{x} y = x \mathrm{sin} x.$$

Multiply both sides of the equation by $e^{-\int \frac{1}{x} dx} = \frac{1}{x}$. Hence we get the equation

$$\frac{1}{x} y' - \frac{1}{x^2} y = \mathrm{sin} x \Rightarrow \frac{1}{x} y = \int \mathrm{sin} x dx = -\mathrm{cos} x + Cx.$$

From the initial condition $y(\pi) = 0$ and the above solution of y, one can easily compute $C = -1$. Thus

$$y = -x \mathrm{cos} x - x.$$

(b) Compute the derivative of the equation, one can deduce the following differential equation with initial condition:

$$f'(x) = \frac{1}{(xf(x))^2}, \ f(2) = 3.$$

Set $f(x) = y$, we have

$$y' = \frac{1}{x^2 y^2}, \quad y(2) = 3.$$

One can see that the equation is a separable equation

$$y^2 y' = \frac{1}{x^2} \Rightarrow \int y^2 dy = \int \frac{1}{x^2} dx \Rightarrow \frac{1}{3} y^3 = \frac{-1}{x} + C.$$

Since $y(2) = 3$, one can compute that $C = \frac{19}{2}$. Thus

$$f(x) = \sqrt[3]{\frac{-3}{x} + \frac{19}{2}}.$$

7. Explain why each geometric series is convergent. Use the formula $\sum_{n=1}^{\infty} a \cdot r^{n-1} = \frac{a}{1-r}$ to find the sum.

(a) (2%) $8 + 0.8 + 0.08 + 0.008 + 0.0008 + \cdots$

(b) (2%) $\sum_{n=0}^{\infty} \pi^n e^{-2n}$

Solution:

(a)

$$\sum_{n=0}^{\infty} 8 \cdot \left(\frac{1}{10}\right)^n = 8 \cdot \frac{1}{1 - 1/10} = \frac{80}{9}.$$

(b)

$$\sum_{n=0}^{\infty} \pi^n e^{-2n} = \sum_{n=0}^{\infty} \left(\frac{\pi}{e^2}\right)^n = \frac{1}{1 - \frac{\pi}{e^2}} = \frac{e^2}{e^2 - \pi}.$$

8. Use the given formula (the definition of Taylor series) to find the Taylor series for the function $f(x) = e^{2x}$ centered at $a = 3$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(a) (4%) Find the first four nonzero terms.

(b) (2%) Find an expression for the n-th term, then write the Taylor series in the form $\sum_{n=0}^{\infty}$ ______________.

Solution:

$$f(3) = e^6, f'(3) = 2e^6, f''(3) = 4e^6, f'''(3) = 8e^6, \ldots, f^{(n)}(3) = 2^n e^6, \cdots$$

The first four nonzero terms

$$e^6 + 2e^6(x-3) + 4e^6 \frac{(x-3)^2}{2!} + 8e^6 \frac{(x-3)^3}{3!} + \cdots$$

$$e^6 + 2e^6(x-3) + 2e^6(x-3)^2 + \frac{4}{3}e^6(x-3)^3 + \cdots$$

General form

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n \cdot e^6}{n!}(x-3)^n$$

□

Grading:

- If student tries to use the formula $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then they can get at most (2%) if they can find the Maclaurin series for e^{2x} correctly. However, if the student successfully apply a substitution $x \to (x-3)$, they can get full credit.
- (-1%) for each minor mistake. (-2%) for each major mistake. Be generous, students are not familiar with series.

9. Use the given formula to find the Maclaurin series of the function.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

(a) (5%) $f(x) = \int_0^x \frac{\cos(\sqrt{t}) - 1}{t}\, dt$

(b) (5%) $g(x) = \dfrac{x^2}{(1+2x)^2}$

Solution:

(a)

$$\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$$

$$\cos\sqrt{t} = 1 - \frac{t}{2} + \frac{t^2}{24} - \frac{t^3}{720} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(2n)!}$$

$$\cos\sqrt{t} - 1 = -\frac{t}{2} + \frac{t^2}{24} - \frac{t^3}{720} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{(2n)!}$$

$$\frac{\cos\sqrt{t} - 1}{t} = -\frac{1}{2} + \frac{t}{24} - \frac{t^2}{720} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n t^{n-1}}{(2n)!}$$

$$\int \frac{\cos\sqrt{t} - 1}{t}\, dt = \left(-\frac{t}{2} + \frac{t^2}{48} - \frac{t^3}{2160} + \cdots\right) + C = \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n \cdot (2n)!} + C$$

$$\int_0^x \frac{\cos\sqrt{t} - 1}{t}\, dt = -\frac{x}{2} + \frac{x^2}{48} - \frac{x^3}{2160} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot (2n)!}$$

(b)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\frac{1}{(1+2x)^2} = 0 + 1 - 4x + 12x^2 + \cdots = \sum_{n=0}^{\infty} n(-1)^{n-1} 2^{n-1} x^{n-1}$$

$$\frac{x^2}{(1+2x)^2} = 0 + x^2 - 4x^3 + 12x^4 + \cdots = \sum_{n=0}^{\infty} n(-1)^{n-1} 2^{n-1} x^{n+1}$$

□

Grading:

- Any successful step is worth (1%).
- If the student can write out a few nonzero terms of the final answer, it is worth (2%).
- (-1%) for each minor mistake. (-2%) for each major mistake. Be generous, students are not familiar with series.