**112 模組11-14班 微積分2 期考解答和評分標準**

1. (27%) Calculate the following integrals.

(a) (8%) $\displaystyle\int_{-1}^{1}(x+1)^2e^{2x}\,\mathrm{d}x.$ (b) (7%) $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{\sqrt{4x^2-1}}\,\mathrm{d}x.$ (c) (12%) $\displaystyle\int_{0}^{2}\frac{x^3-x^2-4x+12}{(x+2)^2(x^2+4)}\,\mathrm{d}x.$

**Solution:**

(a) We use integration by parts twice in which we always integrate exponential functions and differentiate polynomials.

$$\int_{-1}^{1}(x+1)^2e^{2x}\,dx = (x+1)^2\frac{1}{2}e^{2x}\Big|_{x=-1}^{x=1} - \int_{-1}^{1}2(x+1)\frac{1}{2}e^{2x}\,dx \qquad \text{(2 pts for integration by parts)}$$

$$= 2e^2 - 0 - \int_{-1}^{1}(x+1)e^{2x}\,dx = 2e^2 - \int_{-1}^{1}(x+1)e^{2x}\,dx \qquad \text{(1 pt for evaluating } (x+1)^2\frac{1}{2}e^{2x}\Big|_{x=-1}^{x=1}\text{)}$$

$$= 2e^2 - (x+1)\frac{1}{2}e^{2x}\Big|_{x=-1}^{x=1} + \int_{-1}^{1}\frac{1}{2}e^{2x}\,dx \qquad \text{(2 pts for intrgation by parts)}$$

$$= 2e^2 - e^2 + \frac{1}{2}\int_{-1}^{1}e^{2x}\,dx = e^2 + \frac{1}{2}\int_{-1}^{1}e^{2x}\,dx \qquad \text{(1 pt for evaluating } (x+1)\frac{1}{2}e^{2x}\Big|_{x=-1}^{x=1}\text{)}$$

$$= e^2 + \frac{1}{4}(e^2 - e^{-2}) = \frac{5}{4}e^2 - \frac{1}{4}e^{-2} \qquad \text{(2 pts for integrating } e^{2x} \text{ and the final answer)}$$

(b) Let $2x = \sec\theta$ where $0 \le \theta < \frac{1}{2}\pi,\ \pi \le \theta < \frac{3}{2}\theta$. Then $dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$. (2 pts)

$$\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{\sqrt{4x^2-1}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{\tan\theta}\frac{1}{2}\sec\theta\tan\theta\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{2}\sec\theta\,d\theta$$

$$= \frac{1}{2}\ln|\sec\theta + \tan\theta|\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} = \frac{1}{2}(\ln|2+\sqrt{3}| - \ln|\sqrt{2}+1|).$$

(3 pts total for the trigonometric substitution: 2 pts for the integrand $\frac{1}{2}\sec\theta$ and 1 pt for the upper and lower bounds, $\frac{\pi}{3}, \frac{\pi}{4}$.
1 pt for $\sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$.
1 pt for the final answer.)

(c) We write $\dfrac{x^3-x^2-4x+12}{(x+2)^2(x^2+4)}$ as the sum of partial fractions.

$$\frac{x^3-x^2-4x+12}{(x+2)^2(x^2+4)} = \frac{a}{x+2} + \frac{b}{(x+2)^2} + \frac{cx+d}{x^2+4}. \qquad \text{(2 pts for correct and complete partial fractions )}$$

Hence,

$$x^3 - x^2 - 4x + 12 = a(x+2)(x^2+4) + b(x^2+4) + (cx+d)(x+2)^2.$$

Let $x = -2$. We obtain

$$8 = b(4+4) \implies b = 1.$$

Compare coefficients of $x^3, x^2, x$ and we have

$$\begin{aligned} x^3 &: \quad 1 = a + c \\ x^2 &: \quad -1 = 2a + b + 4c + d \\ x &: \quad -4 = 4a + 4c + 4d \end{aligned}$$

Therefore, $a = 2, c = -1, d = -2$, i.e.

$$\frac{x^3-x^2-4x+12}{(x+2)^2(x^2+4)} = \frac{2}{x+2} + \frac{1}{(x+2)^2} + \frac{-x-2}{x^2+4}. \qquad \text{(4 pts for constants } a, b, c, d\text{)}$$

Thus,

$$\int_0^2 \frac{x^3 - x^2 - 4x + 12}{(x+2)^2(x^2+4)}\,dx = \int_0^2 \frac{2}{x+2} + \frac{1}{(x+2)^2} + \frac{-x-2}{x^2+4}\,dx$$

$$= \left(2\ln|x+2| - \frac{1}{x+2} - \frac{1}{2}\ln(x^2+4) - \arctan(\frac{x}{2})\right)\Big|_{x=0}^{x=2} = \frac{3}{2}\ln 2 + \frac{1}{4} - \frac{\pi}{4}.$$

(1 pt for $\int \frac{2}{x+2}\,dx = 2\ln|x+2| + C$. 1 pt for $\int \frac{1}{(x+2)^2}\,dx = -\frac{1}{x+2} + C$.

2 pts for $\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4) + C$. 2 pts for $\int \frac{2}{x^2+4}\,dx = \arctan(\frac{x}{2}) + C$.

If students only compute the indefinite integral without computing the definite integral or obtain wrong final answer $\frac{3}{2}\ln 2 + \frac{1}{4} - \frac{\pi}{4}$, they have 0.5 point deducted. )

2. (12%) Let $g(x) = \int_x^{x+1} e^{-t^2}\,dt$ for all real numbers $x$.

(a) (3%) Find $g'(x)$ for all real numbers $x$.

(b) (4%) Find the interval(s) of increase and interval(s) of decrease of the function $g(x)$, and then find a real number $x_0$ such that $g(x)$ attains its absolute maximum at $x = x_0$.

(c) (2%) Show that $g(x) \le 1$ for all real numbers $x$.

(d) (3%) Find $\lim_{x\to\infty} g(x)$.

**Solution:**

(a) We have $g(x) = \int_0^{x+1} e^{-t^2}\,dt - \int_0^x e^{-t^2}\,dt$ (1%), so $g'(x) = e^{-(x+1)^2} - e^{-x^2}$ (2%). □

(b) By (a), $g'(x) = e^{-x^2}(e^{-2x-1} - 1)$ for all $x \in \mathbb{R}$, so $g'(x) > 0$ for $x < -1/2$, $g'(-1/2) = 0$ and $g'(x) < 0$ for $x > -1/2$. This shows that $g(x)$ is (strictly) increasing on $x \le -1/2$ and is (strictly) decreasing on $x \ge -1/2$, so $g(x) \le g(-1/2)$ for all $x \in \mathbb{R}$, and we may thus take $x_0 = -1/2$. □

Grading: 1% for $g'(-1/2) = 0$, 1% for the interval of increase of $g(x)$, 1% for the interval of decrease of $g(x)$, and 1% for $x_0 = -1/2$. For the interval of increase $x \le -1/2$, one can also write it as $(-\infty, -1/2]$, and do **not** deduct points if one does not include $x = -1/2$ in the interval and write the interval of increase as $x < -1/2$ or $(-\infty, -1/2)$. Apply the same grading criteria for the interval of decrease.

(c) **(Method 1)** Since $e^{-t^2} \le 1$ for all $t \in \mathbb{R}$ (1%), we have $g(x) \le \int_x^{x+1} 1\,dt = 1$ for all $x \in \mathbb{R}$ (1%). □

**(Method 2)** In (b) we have found that $g(x) \le g(-1/2)$ for all $x \in \mathbb{R}$ (1%). Since $g(-1/2) = \int_{-1/2}^{1/2} e^{-t^2}\,dt \le \int_{-1/2}^{1/2} 1\,dt = 1$, we conclude that $g(x) \le 1$ for all $x \in \mathbb{R}$ (1%). □

(d) **(Method 1)** For $x \le t \le x+1$ with $x > 0$, we have $0 \le e^{-t^2} \le e^{-x^2}$, so $0 \le g(x) \le \int_x^{x+1} e^{-x^2}\,dt = e^{-x^2}$ for all $x > 0$ (1%). Since $\lim_{x\to\infty} e^{-x^2} = 0$ (1%), we conclude (by squeezing) that $\lim_{x\to\infty} g(x) = 0$ (1%). □

**(Method 2)** By the mean value theorem for integrals, for every $x \in \mathbb{R}$ we have $g(x) = e^{-t(x)^2}$ for some $x \le t(x) \le x+1$ (1%). As $x \to \infty$, we have $t(x) \to \infty$ and hence $g(x) = e^{-t(x)^2} \to 0$ (2%). □

**(Method 3)** As $\int_0^\infty e^{-t^2}\,dt$ is convergent (2%), we see that

$$g(x) = \int_0^{x+1} e^{-t^2}\,dt - \int_0^x e^{-t^2}\,dt \xrightarrow{x\to\infty} \int_0^\infty e^{-t^2}\,dt - \int_0^\infty e^{-t^2}\,dt = 0 \text{ (1\%)}.$$ □

*Remark.* The fact that $\lim_{x\to\infty} g'(x) = 0$ is **not** sufficient in itself to explain why $\lim_{x\to\infty} g(x) = 0$.

3. (14%) In the $xy$-plane, consider two parabolas

$$C_1 : y = \frac{1}{10}x(x-7) \quad \text{and} \quad C_2 : x = \frac{1}{4}y^2 + \frac{7}{4},$$

and let $L$ be the line $x = 4$. Observe that the point $P = (2, -1)$ lies on both $C_1$ and $C_2$, and that the point $Q = (4, 3)$ lies on both $C_2$ and $L$. (See Figure 1.)

(a) (7%) Find the area of the region bounded by $C_1$, $L$ and the line segment $PQ$.

(b) (7%) Find the area of the region bounded by $C_2$ and the line segment $PQ$.

**Note:** parabola = 拋物線; line segment = 線段.

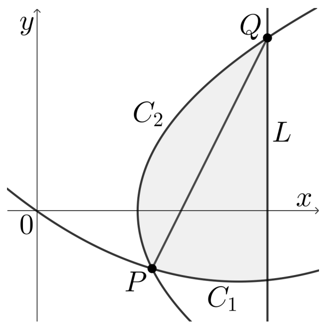


Figure 1

**Solution:**

(a) **(Method 1)** The equation of the line $PQ$ is $y = 2x - 5$ (1%), so the required area is

$$\int_2^4 (2x-5) - \frac{1}{10}x(x-7)\,dx \text{ (2\%)} = \left[(x^2 - 5x) - \frac{1}{10}\left(\frac{1}{3}x^3 - \frac{7}{2}x^2\right)\right]_{x=2}^{x=4} \text{ (2\%)} = \frac{13}{3} \text{ (2\%)}. \qquad \square$$

**(Method 2)** Let $R$ be the intersection of $L$ and $C_1$, so that we can divide the region by the line $PR$ into two parts. We have $R = (4, -6/5)$ (1%), so the area of $\triangle PQR$ is $\frac{1}{2} \cdot \frac{21}{5} \cdot 2 = \frac{21}{5}$ (1%). Using Archimedes' quadrature of parabola, the area $A$ of the region bounded by $C_1$ and the line $PR$ is $\frac{4}{3} \cdot$ (Area of $\triangle PRS$) where $S$ is the intersection of $C_1$ and the vertical line passing through the midpoint of the line segment $PR$ (2% for correctly stating the formula; no proof is needed). As $S = (3, -6/5)$ (1%), the area of $\triangle PRS$ is $\frac{1}{2} \cdot 1 \cdot \frac{1}{5} = \frac{1}{10}$, so $A = \frac{4}{3} \cdot \frac{1}{10} = \frac{2}{15}$ (1%). Thus the required area is $\frac{21}{5} + \frac{2}{15} = \frac{13}{3}$ (1%). $\square$

(b) **(Method 1)** The equation of the line $PQ$ is also $x = \frac{1}{2}y + \frac{5}{2}$ (1%). So the required area is

$$\int_{-1}^3 \left(\frac{1}{2}y + \frac{5}{2}\right) - \left(\frac{1}{4}y^2 + \frac{7}{4}\right) dy \text{ (2\%)} = \left[\left(\frac{1}{4}y^2 + \frac{5}{2}y\right) - \left(\frac{1}{12}y^3 + \frac{7}{4}y\right)\right]_{y=-1}^{y=3} \text{ (2\%)} = \frac{8}{3} \text{ (2\%)}. \qquad \square$$

**(Method 2)** One can divide the region by the line $x = 2$ (the vertical line passing through $P$) into two parts and rewrite the equation of $C_2$ as $y = \pm\sqrt{4x-7}$ (1%) to see that the required area is

$$\int_{7/4}^2 2\sqrt{4x-7}\,dx + \int_2^4 \sqrt{4x-7} - (2x-5)\,dx \text{ (2\%)}$$
$$= \left[\frac{1}{3}(4x-7)^{3/2}\right]_{x=7/4}^{x=2} + \left[\frac{1}{6}(4x-7)^{3/2} - (x^2 - 5x)\right]_{x=2}^{x=4} \text{ (2\%)} = \frac{1}{3} + \frac{7}{3} = \frac{8}{3} \text{ (2\%)}. \qquad \square$$

**(Method 3)** Using Archimedes' quadrature of parabola, the required area is $\frac{4}{3} \cdot$ (Area of $\triangle PQT$) where $T$ is the intersection of $C_2$ and the horizontal line passing through the midpoint of the line segment $PQ$ (2% for correctly stating the formula; no proof is needed). We have $T = (2, 1)$ (1%), so the area of $\triangle PQT$ is $\frac{1}{2} \cdot 2 \cdot 2 = 2$ (2%) and hence the required area is $\frac{4}{3} \cdot 2 = \frac{8}{3}$ (2%). $\square$

4. (a) (6%) Suppose that $D = D(p)$ is the demand function of a commodity. The elasticity of demand at price $p$ is defined as

$$\epsilon(p) = \frac{dD}{dp}\frac{p}{D}.$$

Find all demand functions $D(p)$ that have constant elasticity $\epsilon(p) = \gamma < 0$.

(b) (10%) Assume the unit price of a commodity, $p$, is a function of time $t$ and its rate of change is the difference between the demand ($D$) and the supply ($S$) expressed as

$$\frac{dp}{dt} = D - S.$$

Let's consider that both demand and supply depend on the unit price ($p$) and time ($t$). In particular,

$$D = 50 - 2p + e^{-5t},\ S = 5 + 3p - 4e^{-5t}.$$

Given the initial condition $p(0) = 5$, solve $p(t)$ and find $\lim_{t\to\infty} p(t)$.

**Solution:**

(a) $\epsilon(p) = \dfrac{dD}{dp}\dfrac{p}{D(p)} = \gamma$ implies that

$$\int \frac{dD}{D} = \int \gamma\frac{dp}{p} \implies \ln(D) = \gamma\ln(p) + C$$

which implies that $D(p) = Ap^{\gamma}$ for some constant $A > 0$.

(2 pts for writing the differential equation as $\int \frac{dD}{D} = \int \gamma\frac{dp}{p}$.
3 pts for $\ln(D) = \gamma\ln(p) + C$.
1 pt for the final answer, $D(p) = Ap^{\gamma}$.)

(b) We have the differential equation

$$\frac{dp}{dt} = 45 - 5p + 5e^{-5t} \implies \frac{dp}{dt} + 5p = 45 + 5e^{-5t}. \qquad \text{(2 pts)}$$

Choose the integrating factor $I(t) = e^{5t}$. (2 pts)
Multiply the equation with $I(t)$ and we obtain

$$(e^{5t}p(t))' = 45e^{5t} + 5. \qquad \text{(2 pts)}$$

Hence

$$e^{5t}p(t) = 9e^{5t} + 5t + C \implies p(t) = 9 + 5te^{-5t} + Ce^{-5t} \quad \text{(2 pts)}$$

Because $p(0) = 5$, we obtain that $C = -4$. (1 pt for $C$)
Therefore $p(t) = 9 + 5te^{-5t} - 4e^{-5t}$ and $\lim_{t\to\infty} p(t) = 9$.
(1 pt for $\lim_{t\to\infty} p(t)$.)

5. (13%) Let $X$ be the length (in cm) of a randomly selected metal rod produced by a factory. Suppose $X$ is a continuous random variable whose **probability distribution function** is given by

$$F(x) = P(X \le x) = \begin{cases} a - be^{-0.2x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}, \text{ where } F(0) = 0.$$

(a) (2%) Find the values of $a$ and $b$.

(b) (3%) Find a probability density function $f(x)$ of the random variable $X$.

(c) (8%) Selling any metal rods of lengths shorter than 1 cm will yield a loss. It is known that the (net) profit $\Pi$ of selling a rod of $x$ cm is given by

$$\Pi(x) = \begin{cases} -2 & \text{if } x < 1 \\ 2x & \text{if } x \ge 1 \end{cases}.$$

Find the expected profit per rod which is $\int_0^\infty \Pi(x) f(x)\,\mathrm{d}x$.

**Solution:**

(a) Since $\lim_{x\to\infty} F(x) = 1$, we obtain $a = 1$.
By using the given condition $F(0) = 0$, we have $a - b = 0$.
Therefore, we deduce that $a = b = 1$.

Grading scheme.

- (1M) Mentioning $\lim_{x\to\infty} F(x) = 1$.
- (0.5M) Correct value of $a$
- (0.5M) Correct value of $b$

**Remark.** Partial credit : 0.5M for those who **only** manage to obtain $a = b$ by using the given condition. In both (a),(b), we may deduct marks from students with correct answers but incorrect and/or incomplete reasonings.

**Graders' comments.** Many students confuse distribution functions as density functions and end up 'integrating $F(x)$' : this results in a massive deduction (not only in this part).

(b) By using the Fundamental Theorem of Calculus, for $x \ne 0$, we have $f(x) = F'(x)$. Therefore, we may take, for example,

$$f(x) = F'(x) = \begin{cases} 0.2e^{-0.2x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Grading scheme.

- (2M) Correct density for $x > 0$ (partial credits are available as long as students demonstrate the knowledge that $F'(x) = f(x)$.)
- (1M) Correct density for $x < 0$

**Graders' comments.** The computations are poor. Many candidates could not distinguish the difference between between the derivative and the integral of $e^{-kx}$ and some have forgotten that the derivative of a constant equals 0. Some have forgotten to mention the density for $x < 0$.

(c) The expected profit is

$$\int_0^\infty \Pi(x) f(x)\,\mathrm{d}x = \int_0^1 (-2)0.2e^{-0.2x}\,\mathrm{d}x + \int_1^\infty (2x)0.2e^{-0.2x}\,\mathrm{d}x$$

For the first integral, we have

$$\int_0^1 (-0.4e^{-0.2x})\,dx = \left[2e^{-0.2x}\right]_0^1 = 2e^{-0.2} - 2$$

For the second integral, we have

$$\int_1^{\infty} (2x)0.2e^{-0.2x}\,\mathrm{d}x \overset{\text{def}}{=} \lim_{t\to\infty} \int_1^t 0.4xe^{-0.2x}\,\mathrm{d}x$$
$$= \lim_{t\to\infty} \left[-2xe^{-0.2x} - 10e^{-0.2x}\right]_1^t$$
$$= \lim_{t\to\infty} (-2te^{-0.2t} - 10e^{-0.2t}) + (2e^{-0.2} + 10e^{-0.2})$$
$$= 12e^{-0.2}$$

where we note that $\lim_{t\to\infty} \frac{t}{e^{0.2t}} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to\infty} \frac{1}{0.2e^{0.2t}} = 0$ by L'Hospital's rule.

Hence, the expected profit equals $\int_0^{\infty} \Pi(x)f(x)\,\mathrm{d}x = 14e^{-0.2} - 2$.

Grading scheme.

- (1M) Split the given integral as two explicit integrals on $[0,1]$ and on $[1,\infty)$ respectively.
- (2M) First integral : correct answer
- (1M) Second integral : correct definition as improper integral
- (2M) Second integral : correct antiderivative of $xe^{-0.2x}$
- (1M) Verify, with derivation, the limit $\lim_{t\to\infty} te^{-0.2t} = 0$ by L'Hopsital's rule
- (1M) Second integral : correct answer

**Remark.** Students with clear misconceptions in distinguishing density/distribution functions or those who derive wildly incorrect density functions in their computations receive very few credits.

**Graders' comments.** Firstly, (although no points have been taken away for this) many candidates think that the given integral is improper at $x = 1$ and end up with some unnecessarily complicated calculations. Secondly, again, the computations have been poor : the integral of $e^{-kx}$ seems particularly difficult to many candidates. Another issues are basic arithmetic : we have seen many elementary errors in signs ($\pm$) and/or computations in decimals/fractions. Some particularly serious mistakes include : forget to integrate and write '$\int_a^b g(x)\,dx = g(b) - g(a)$'.

6. (18%) Let $\alpha, \beta \in (0,1)$ be two parameters. Consider the function $g(x;\alpha,\beta) = x^{\alpha-1}(1-x)^{\beta-1}$ for $x \in (0,1)$.

(a) (i) (4%) Show that the integral (always positive)

$$B(\alpha,\beta) = \int_0^1 g(x;\alpha,\beta)\,\mathrm{d}x = \lim_{t\to 0^+}\int_t^{\frac{1}{2}} x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x + \lim_{s\to 1^-}\int_{\frac{1}{2}}^{s} x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x \quad \text{is convergent.}$$

(ii) (2%) Show that $f(x;\alpha,\beta) = B(\alpha,\beta)^{-1}g(x;\alpha,\beta)$ defines a probability density function on $(0,1)$, which is called *Beta distribution*.

(b) (i) (6%) Compute $B(\frac{1}{2},\frac{1}{2}) = \int_0^1 \frac{1}{\sqrt{x(1-x)}}\mathrm{d}x$.

(ii) (6%) Let $X$ be the random variable having probability density function $f(x;\frac{1}{2},\frac{1}{2})$. Compute the mean (or expectation) $E[X] = \int_0^1 x f(x;\frac{1}{2},\frac{1}{2})\,\mathrm{d}x$.

**Solution:**

(a) (i) Since $(1-x)^{\beta-1}$ is bounded near 0 and $x^{\alpha-1}$ is bounded near 1, the convergence of the integral $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ near 0 is determined by $x^{\alpha-1}$ and near 1 is determined by $(1-x)^{\beta-1}$. Moreover, since $\alpha,\beta \in (0,1)$, $x^{\alpha-1}$ is integrable near 0 and $(1-x)^{\beta-1}$ is integrable near 1,

$$B(\alpha,\beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx < \infty. \text{ (4pt)}$$

(i) Define $f(x;\alpha,\beta) = B(\alpha,\beta)^{-1}g(x;\alpha,\beta)$ for all $x \in (0,1)$. Then $f(x;\alpha,\beta) \geq 0$ and

$$\int_0^1 f(x;\alpha,\beta)dx = B(\alpha,\beta)^{-1}\int_0^1 g(x;\alpha,\beta)dx = 1.$$

Hence $f(x;\alpha,\beta)$ defines a probability density function on $(0,1)$. (2pt)

(b) Note that

$$B(\frac{1}{2},\frac{1}{2}) = \int_0^1 \frac{1}{\sqrt{x(1-x)}}dx = \lim_{a\to 0^+;b\to 1^-}\int_a^b \frac{1}{\sqrt{x(1-x)}}dx.$$

First, we evaluate the indefinite integral

$$\int \frac{1}{\sqrt{x(1-x)}}dx = \int \frac{1}{\sqrt{(1/2)^2-(x-1/2)^2}}dx \overset{y=x-1/2}{=} \int \frac{1}{\sqrt{(1/2)^2-y^2}}dy. \text{ (1\%)}$$

Use trig substitution or the derivative of $\sin^{-1} y$, we obtain that

$$\int \frac{1}{\sqrt{x(1-x)}}dx = \sin^{-1}(2x-1). \text{ (3pt)}$$

So

$$B(\frac{1}{2},\frac{1}{2}) = \lim_{a\to 0^+;b\to 1^-}[\sin^{-1}(2b-1) - \sin^{-1}(2a-1)] = \pi. \text{ (2pt)}$$

(c)

$$\begin{aligned}\int \frac{\sqrt{x}}{\sqrt{1-x}}dx &\overset{y=\sqrt{x}}{=} 2\int \frac{y^2}{\sqrt{1-y^2}}dy \overset{y=\sin\theta}{=} 2\int \sin\theta^2 d\theta \\ &= 2\int \frac{1-\cos(2\theta)}{2}d\theta = 2(\frac{\theta}{2} - \frac{\sin(2\theta)}{4}) \\ &= \theta - \frac{\sin(2\theta)}{2} = \sin^{-1} y - y\sqrt{1-y^2} \\ &= \sin^{-1}(\sqrt{x}) - \sqrt{x(1-x)}.\end{aligned}$$

For $y = \sqrt{x}$ (1pt), for $y = \sin\theta$ (1pt), the answer $\sin^{-1}(\sqrt{x}) - \sqrt{x(1-x)}$ (2pt). Hence

$$\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x}}dx = \frac{\pi}{2}. \text{ (1pt)}$$

and the mean of the random variable with distribution $f(x;\frac{1}{2},\frac{1}{2})$ is

$$\frac{\pi/2}{\pi} = \frac{1}{2}. \text{ (1pt)}$$