**112 模組01-04班 微積分2 期考解答和評分標準**

1. (8%) Show that, for all $a \in \mathbb{R}$,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{a^2}{n+a^2k} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2a^2k}{n^2+a^2k^2}.$$

**Solution:**
We have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{a^2}{n+a^2k} = \lim_{n\to\infty}\frac{a^2}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{a^2}{n}k} = \int_0^{a^2}\frac{1}{1+t}dt$$
$$= \ln(1+a^2).$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2a^2k}{n^2+a^2k^2} = \lim_{n\to\infty}\frac{a}{n}\sum_{k=1}^{n}\frac{2\frac{a}{n}k}{1+(\frac{a}{n}k)^2} = \int_0^{a}\frac{2t}{1+t^2}dt$$
$$= \ln(1+a^2).$$

[Proceed to rewrite the sums into Riemann sums by taking the factor $\frac{1}{n}$ out of the sum-symbol $\sum$: (+2); convert the first sum into a definite integral correctly: (+2); compute the first definite integral correctly: (+1); convert the second sum into a definite integral correctly: (+2); compute the second definite integral correctly: (+1).]

2. (8%) Find **all** functions $f : \mathbb{R} \to \mathbb{R}$ and constants $a \in \mathbb{R}$ that satisfy the equality

$$\int_a^{2x-1} f'(t+1)e^{-t}\,\mathrm{d}t = x^2 - 1.$$

**Solution:**
If $a = 2x - 1$, then the definite integral will be zero. This means that $x = \pm 1$, hence $a = -3$ or $1$.
By taking the derivative with respect to $x$, we get

$$f'(2x-1+1)e^{-(2x-1)}\cdot 2 = 2x$$
$$f'(2x) = xe^{2x-1}$$
$$f'(x) = \frac{1}{2}xe^{x-1}$$
$$f(x) = \frac{1}{2}\int xe^{x-1}\,dx = \frac{1}{2}(xe^{x-1} - e^{x-1}) + C$$

Therefore the final answer is

$$a = -3 \text{ or } 1, \quad f(x) = \frac{e^{x-1}}{2}(x-1) + C \quad \text{for any constant } C.$$

□

Grading:

- Values of $a$ is 3% (Note that they can find the values of $a$ via finding $f$ first, then integrate).
- Using the FTC part 1 correctly is 2%.
- Finding the functions $f$ is 3% (-1% if the student leaves the answer as $f(2x) = \cdots$).
- Each clearly minor mistake is -0.5%, each conceptual mistake is -1%.
- Students can also start the problem with integration by parts, but it will make the process very messy.

3. Find the following integrals.

(a) (5%) $\int \tan^4 x \, dx$. (b) (6%) $\int \frac{1+\sqrt{x}}{1+\sqrt[3]{x}} \, dx$ (c) (8%) $\int_0^1 x^5\sqrt{1-x^4} \, dx$

**Solution:**

(a) Solution 1:

$$\int \tan^4 x \, dx = \int \tan^2 x(\sec^2 - 1) \, dx = \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$$

$$\overset{u=\tan x,\ du=\sec^2 x dx}{=} \frac{1}{3}\tan^3 x - \int (\sec^2 x - 1) \, dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$$

( 1 pt for $\tan^4 x = \tan^2 x(\sec^2 x - 1)$.
2 pts for $\int \tan^2 x \sec^2 x \, dx = \frac{1}{3}\tan^3 x + C$.
2 pts for $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + C$.)

Solution 2:

$$\int \tan^4 x \, dx = \int \frac{\tan^4 x}{\sec^2 x}\sec^2 x \, dx \overset{u=\tan x}{=} \int \frac{u^4}{u^2+1} \, du$$

$$= \int u^2 - 1 + \frac{1}{u^2+1} \, du = \frac{u^3}{3} - u + \arctan(u) + C = \frac{\tan^3 x}{3} - \tan x + x + C.$$

(2 pts for $\tan^4 x = \frac{\tan^4 x}{\sec^2 x}\sec^2 x$ and the substitution $u = \tan x$.
2 pts for integrating $\frac{u^4}{u^2+1}$.
1 pt for substituting $u = \tan x$ and the final answer.)

(b) Let $u = x^{\frac{1}{6}}$. Then $x = u^6$ and $dx = 6u^5 \, du$.

$$\int \frac{1+\sqrt{x}}{1+\sqrt[3]{x}} \, dx = \int \frac{1+u^3}{1+u^2} 6u^5 \, du = 6\int u^6 - u^4 + u^3 + u^2 - u - 1 + \frac{u+1}{u^2+1} \, du$$

$$= 6\left(\frac{u^7}{7} - \frac{u^5}{5} + \frac{u^4}{4} + \frac{u^3}{3} - \frac{u^2}{2} - u + \frac{1}{2}\ln(u^2+1) + \arctan u\right) + C$$

$$= 6\left(\frac{x^{\frac{7}{6}}}{7} - \frac{x^{\frac{5}{6}}}{5} + \frac{x^{\frac{2}{3}}}{4} + \frac{\sqrt{x}}{3} - \frac{\sqrt[3]{x}}{2} - x^{\frac{1}{6}} + \frac{1}{2}\ln(\sqrt[3]{x}+1) + \arctan(x^{\frac{1}{6}})\right) + C$$

(1 pt for choosing $u = x^{\frac{1}{6}}$ and $dx = 6u^5 \, du$.
1 pt for the integrand $6\frac{u^8+u^5}{u^2+1}$.
1 pt for $\frac{u^8+u^5}{u^2+1} = u^6 - u^4 + u^3 + u^2 - u - 1 + \frac{u+1}{u^2+1}$.
2 pts for integrating $\frac{u+1}{u^2+1}$.
1 pt for substituting $u = x^{\frac{1}{6}}$ and the final answer.)

(c) Let $u = x^2$. Then $du = 2x \, dx$.

$$\int_0^1 x^5\sqrt{1-x^4} \, dx = \int_0^1 \frac{1}{2}u^2\sqrt{1-u^2} \, du. \qquad \text{(2 pts)}$$

Let $u = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $du = \cos\theta \, d\theta$.

$$\int_0^1 u^2\sqrt{1-u^2} \, du = \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{8} d\theta = \frac{\pi}{16}.$$

Hence $\int_0^1 x^5\sqrt{1-x^4} \, dx = \frac{1}{2}\int_0^1 u^2\sqrt{1-u^2} \, du = \frac{\pi}{32}$.

(1 pt for choosing $u = \sin\theta$.
1 pt for the integrand $\sin^2\theta\cos^2\theta$.
1 pt for the upper and lower limits for $\theta$, 0 and $\frac{\pi}{2}$.
2 pts for the identity $\sin^2\theta\cos^2\theta = \frac{1-\cos 4\theta}{8}$ or
$\sin^2\theta\cos^2\theta = \frac{1-\cos 2\theta}{2}\frac{1+\cos 2\theta}{2} = \frac{1-\cos^2 2\theta}{4} = \frac{\sin^2 2\theta}{4} = \frac{1-\cos 4\theta}{8}$.
1 pt for the definite integral $\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{8}\, d\theta = \frac{\pi}{16}$.)

4. Let $k$ be a positive constant less than $\frac{\pi}{2}$ and $R_k$ be the region enclosed by the curves $y = 1 + \tan x$ and $y = 1 + \sec x$ between $x = 0$ and $x = k$.

(a) (5%) Consider the solid $S_k$ obtained by rotating the region $R_k$ about the $x$-axis. Find the volume of $S_k$ and find the limit as $k$ approaches $\frac{\pi}{2}$.

(b) (5%) Consider the solid $T_k$ obtained by rotating the region $R_k$ about the $y$-axis. Determine whether the volume of $T_k$ is finite or infinite as $k$ approaches $\frac{\pi}{2}$. (Note: you may not be able to evaluate the exact volume of $T_k$.)

**Solution:**

(a) Improper integral approach.

$$\lim_{k\to(\frac{\pi}{2})^-} \int_0^k \pi(1+\sec x)^2 - \pi(1+\tan x)^2 \, dx$$

$$= \lim_{k\to(\frac{\pi}{2})^-} \pi \int_0^k (2\sec x - 2\tan x + 1) \, dx$$

$$= \lim_{k\to(\frac{\pi}{2})^-} \pi(2\ln|\sec k + \tan k| - 2\ln|\sec k| + k)$$

$$= \frac{\pi^2}{2} + 2\pi \ln 2$$

(b) Improper integral approach.

$$\lim_{k\to(\frac{\pi}{2})^-} \int_0^k 2\pi x(1+\sec x) - 2\pi x(1+\tan x) \, dx$$

Since $x$ is bounded, we can use the inequality

$$2\pi x(1+\sec x) - 2\pi x(1+\tan x) \le \pi^2(\sec x - \tan x)$$

and then use our result in (a) to prove that the volume is finite. □

Grading:

- Formula for each of the volume is 2% (-1% if they are only missing a constant, otherwise all or nothing).
- If they are using an incorrect formula, then grade the rest of the problem strictly. -1% for each mistake or missing step.
- The integral in (a) is 2% and the limit is 1% (so -1% if they didn't write limit).
- There are many different choices of comparison in (b). They can also look at the limit of $\sec x - \tan x$. Formal wording of the comparison theorem is 2%. 1% for evaluating or explaining the convergence or divergence of the integral of the comparison function.
- Each clearly minor mistake is -0.5%, each conceptual mistake is -1%.

5. Let $F_n = \int x^n e^{-x^2}\, dx$ and $I_n = \int_0^\infty x^n e^{-x^2}\, dx$.

(a) (2%) Find $F_1$.

(b) (3%) Show that for any integer $n \geq 2$, $F_n = -\frac{1}{2}x^{n-1}e^{-x^2} + \frac{n-1}{2}F_{n-2}$.

(c) (5%) It is known that $I_0 = \frac{\sqrt{\pi}}{2}$. Find $I_{2n}$ where $n$ is a positive integer.

(d) (5%) Let $p \in \mathbb{R}$ ($p$ is not necessarily an integer anymore). Determine the range of values of $p$ such that the improper integral $I_p$ converges.

**Solution:**

(a)

$$F_1 = \int xe^{-x^2}\, dx \overset{u=x^2,\ du=2x\,dx}{=} \int \frac{1}{2}e^{-u}\, du = -\frac{1}{2}e^{-u} + C = -\frac{1}{2}e^{-x^2} + C.$$

(1 pt for the substitution $u = x^2$.
1 pt for the final answer.)

(b) By integration by parts,

$$F_n = \int x^{n-1}xe^{-x^2}\, dx = x^{n-1}(\frac{-1}{2}e^{-x^2}) + \frac{n-1}{2}\int x^{n-2}e^{-x^2}\, dx = -\frac{1}{2}x^{n-1}e^{-x^2} + \frac{n-1}{2}F_{n-2}.$$

(1 pt for splitting $x^n e^{-x^2}$ as the product of $x^{n-1}$ and $xe^{-x^2}$.
2 pts for integrating by parts and the final formula.)

(c) By the reduction formula in (a), we have

$$\int_0^t x^{2n}e^{-x^2}\, dx = -\frac{1}{2}t^{2n-1}e^{-t^2} + \frac{2n-1}{2}\int_0^t x^{2n-2}e^{-x^2}\, dx.$$

Hence

$$I_{2n} = \lim_{t\to\infty}\int_0^t x^{2n}e^{-x^2}\, dx = -\frac{1}{2}\lim_{t\to\infty} t^{2n-1}e^{-t^2} + \frac{2n-1}{2}\lim_{t\to\infty}\int_0^t x^{2n-2}e^{-x^2}\, dx \quad \text{(2 pts)}.$$

If $I_{2n-2}$ converges, then $\lim_{t\to\infty}\int_0^t x^{2n-2}e^{-x^2}\, dx$ exists and equals $I_{2n-2}$. Moreover, $\lim_{t\to\infty} t^{2n-1}e^{-t^2} = 0$. Hence, if $I_{2n-2}$ converges, then $I_{2n}$ also converges and

$$I_{2n} = \frac{2n-1}{2}I_{2n-2}.$$

Because that $I_0$ converges, by mathematical induction, $I_{2n}$ converges for all positive integer $n$ and

$$I_{2n} = \frac{(2n-1)(2n-3)\cdots 1}{2^n}I_0 = \frac{(2n-1)(2n-3)\cdots 1}{2^n}\frac{\sqrt{\pi}}{2}.$$

(1 pt for deriving that the convergence of $I_0$ implies the convergence of $I_{2n}$ for all positive integer $n$.
2 pts for $I_{2n} = \frac{2n-1}{2}I_{2n-2}$.
2 pts for $I_{2n} = \frac{(2n-1)(2n-3)\cdots 1}{2^n}I_0 = \frac{(2n-1)(2n-3)\cdots 1}{2^n}\frac{\sqrt{\pi}}{2}$.)

(d) If $p < 0$, then $\lim_{x\to 0^+} x^p e^{-x^2} = \infty$. Hence the improper integral $\int_0^\infty x^p e^{-x^2}\, dx$ may be an improper integral of both type I and type II. Therefore, we should write $I_p$ as

$$\int_0^1 x^p e^{-x^2}\, dx + \int_1^\infty x^p e^{-x^2}\, dx$$

and $I_p$ converges if and only if both improper integrals are convergent.

Let's first investigate $\int_1^\infty x^p e^{-x^2}\, dx$. In part (c), we have shown that $I_{2n}$ converges for all positive integer $n$. For any $p \in \mathbf{R}$, we can find a positive integer $n_0$ such that $2n_0 > p$. Then for $x \geq 1$,

$$0 < x^p e^{-x^2} \leq x^{2n_0}e^{-x^2}.$$

Therefore, the convergence of $I_{2n_o}$ implies that $\int_1^\infty x^p e^{-x^2}\,dx$ converges by the comparison theorem.
In conclusion, $\int_1^\infty x^p e^{-x^2}\,dx$ converges for all $p \in \mathbf{R}$.

Now we consider $\int_0^1 x^p e^{-x^2}\,dx$. Note that

$$0 < \frac{1}{e}x^p \le x^p e^{-x^2} \le x^p \qquad \text{for} \quad 0 < x \le 1.$$

Hence by the comparison theorem, $\int_0^1 x^p e^{-x^2}\,dx$ converges if and only if $\int_0^1 x^p\,dx$ converges. And we know that $\int_0^1 x^p\,dx$ converges if and only if $p > -1$. Therefore, $\int_0^1 x^p e^{-x^2}\,dx$ is convergent if and only if $p > -1$.

As a result, $I_p$ converges if and only if $p > -1$.

(2 pts for the convergence of $\int_1^\infty x^p e^{-x^2}\,dx$ for all $p$.
3 pts for the convergence of $\int_0^1 x^p e^{-x^2}\,dx$ if $p > -1$.)

6. Let $f$ be a continuous function on $\mathbb{R}$ such that $f(x) \geq 0$ for all $x$. Suppose there exists a constant $T > 0$ such that

$$f(x+T) = f(x) \text{ for all } x \in \mathbb{R}.$$

(a) (3%) Prove that $\int_{a+kT}^{b+kT} e^{-x} f(x)\,\mathrm{d}x = e^{-kT} \int_a^b e^{-x} f(x)\,\mathrm{d}x$ for any positive integer $k$.

(b) (2%) Let $I_n = \int_0^{nT} e^{-x} f(x)\,\mathrm{d}x$. Using (a), find $I_n$ in terms of $I_1$.

(c) (3%) Let $t$ be a positive number and $n$ be an integer such that $nT \leq t \leq (n+1)T$. Prove that

$$\frac{1-e^{-nT}}{1-e^{-T}} I_1 \leq \int_0^t e^{-x} f(x)\,\mathrm{d}x \leq \frac{1-e^{-(n+1)T}}{1-e^{-T}} I_1.$$

(d) (2%) Use (c) to deduce that $\int_0^\infty e^{-x} f(x)\mathrm{d}x$ converges and express it in terms of $T$ and $I_1$.

(e) (6%) Use (d) to evaluate $\int_0^\infty e^{-x} |\sin x|\,\mathrm{d}x$.

**Solution:**

(a) Let $u = x - kT$. Then

$$\begin{aligned}\int_{a+kT}^{b+kT} e^{-x} f(x)\,\mathrm{d}x &= \int_a^b e^{-u-kT} f(u+kT)\,\mathrm{d}x \\ &= e^{-kT} \int_a^b f(u+kT)\,\mathrm{d}x \\ &= e^{-kT} \int_a^b f(u)\,\mathrm{d}u\end{aligned}$$

Grading scheme.

- (1M) Use the substitution $u = x - kT$.
- (1M) Transform the given integral
- (1M) Overall coherence of the argument

(b) By using (a), for any $a \in \mathbb{R}$, we have

$$\int_{aT}^{(a+1)T} e^{-x} f(x)\,\mathrm{d}x = e^{-aT} \int_0^T e^{-x} f(x) = e^{-aT} \cdot I_1.$$

As a result,

$$\begin{aligned}I_n &= \int_0^T e^{-x} f(x)dx + \int_T^{2T} e^{-x} f(x)dx + \int_{2T}^{3T} e^{-x} f(x)\,\mathrm{d}x \cdots + \int_{(n-1)T}^{nT} e^{-x} f(x)dx \\ &= I_1 + e^{-T} I_1 + e^{-2T} I_1 + \cdots + e^{-(n-1)T} I_1 \\ &= I_1 (1 + e^{-T} + \cdots + e^{-(n-1)T})\end{aligned}$$

Grading scheme.

- (1M) Use (a) correctly on an integral of the form $\int_{aT}^{(a+1)T} e^{-x} f(x)dx$
- (1M) Correct answer

(c) Note that $e^{-x} f(x) \geq 0$. For $nT \leq t \leq (n+1)T$, we have $I_n \leq \int_0^t e^{-x} f(x)dx \leq I_{n+1}$.

By using (b) and the formula of a geometric sum, we have $I_n = \dfrac{1-e^{-nT}}{1-e^{-T}} \cdot I_1$.
Combining these imply the desired inequality,

$$\frac{1-e^{-nT}}{1-e^{-T}} I_1 \leq \int_0^t e^{-x} f(x)\,\mathrm{d}x \leq \frac{1-e^{-(n+1)T}}{1-e^{-T}} I_1.$$

Grading scheme.

- (1M) Mention $e^{-x} f(x) \geq 0$
- (1M) Use monotonicity of integrals
- (1M) Compute the geometric sum $1 + e^{-T} + \cdot + e^{-(n-1)T}$ (Give this credit to those who evaluated the sum in (b).)

(d) Let's take $n$ (and hence $t$) to $\infty$ in the inequality of (c).

As $\lim_{n\to\infty} \frac{1-e^{-nT}}{1-e^{-T}} I_1 = \lim_{n\to\infty} \frac{1-e^{-(n+1)T}}{1-e^{-T}} I_1 = \frac{1}{1-e^{-T}} I_1$. By squeeze theorem, we have

$$\int_0^\infty e^{-x} f(x)\,\mathrm{d}x = \frac{1}{1-e^{-T}} I_1$$

Grading scheme.

- (1M) Use Squeeze Theorem argument
- (1M) Correct answer

(e) Let $f(x) = |\sin(x)|$. Note that in this case, we can take $T = \pi$. By using (d), we have

$$\int_0^\infty e^{-x} |\sin(x)|\,\mathrm{d}x = \frac{1}{1-e^{-\pi}} \int_0^\pi e^{-x} \sin(x)\,\mathrm{d}x$$

By integration-by-part twice, we have

$$\int e^{-x} \sin(x) dx = -e^{-x}(\sin(x) + \cos x) - \int e^{-x} \sin(x) dx \Rightarrow \int e^{-x} \sin(x) dx = -\frac{1}{2} e^{-x}(\sin x + \cos x) + C.$$

Therefore, $\int_0^\pi e^{-x} \sin(x)\, dx = \frac{1+e^{-\pi}}{2}$. Hence,

$$\int_0^\infty e^{-x} |\sin(x)|\,\mathrm{d}x = \frac{1+e^{-\pi}}{2(1-e^{-\pi})}$$

Grading scheme.

- (1M) Identifying the correct $|f(x)|$ and $T$
- (1M) Applying the formula in (d) correctly
- (3M) Evaluate the indefinite integral $\int e^{-x} \sin(x)\,\mathrm{d}x$ (Partial credits available for minor errors)
- (1M) Correct answer

7. A patient takes 100 mg of a certain drug, which is gradually absorbed by the body and then eventually excreted out of the body. After time $t$, let

- $x(t)$ mg be the amount of drug still unabsorbed,
- $y(t)$ mg be the amount of drug absorbed and remained in the body,
- $z(t)$ mg be the amount of drug excreted out of the body.

It is known that the total amount of drug $x(t) + y(t) + z(t) = 100$ is a constant over time. Moreover, $x(0) = 100$, $y(0) = z(0) = 0$.

(a) (3%) It is known that $\frac{dx}{dt} = -0.4x$. Find $x(t)$.

(b) (7%) It is known that $\frac{dz}{dt} = 0.08y$. Derive a first order linear equation for $y = y(t)$. Hence, solve for $y(t)$.

(c) (2%) Hence, find the time when the amount of drug absorbed and remained in the body is maximized. (You don't need to justify maximality.)

**Solution:**

(a) Given $x'(t) = -0.4x$. By separation of variables, we have

$$\int \frac{1}{x} dx = \int -0.4\, dt \Rightarrow x = Ae^{-0.4t}.$$

As $x(0) = 100$, we have $A = 100$. Thus, $x(t) = 100e^{-0.4t}$.

Grading scheme.

- (2M) Writing down $x = Ae^{-0.4t}$ (Partial credits available)
- (1M) Showing that $A = 100$.

(b) Given $z'(t) = 0.08y$. Moreover, as $x'(t) + y'(t) + z'(t) = 0$, we have

$$-40e^{-0.4t} + y'(t) + 0.08y = 0 \Rightarrow y' + 0.08y = 40e^{-0.4t}.$$

An integrating factor is given by $e^{0.08t}$. Multiplying this to both sides of the above equation and integrating gives

$$e^{0.08t}y = \int 40e^{-0.32t}\, dt = -125e^{-0.32t} + C$$

As $y(0) = 0$, we have $C = 125$. Thus, $y(t) = -125e^{-0.4t} + 125e^{-0.08t}$.

Grading scheme.

- (1M) Writing down $x' + y' + z' = 0$.
- (1M+1M) Obtaining the correct $p(t), q(t)$ of the first order equation $y' + p(t) \cdot y = q(t)$.
- (1M) Correct integrating factor
- (2M) Correct general solution for $y(t)$
- (1M) Correct constant $C$

(c) Set $y'(t) = 0$. We have $-10e^{-0.4t} + 10e^{-0.08t} = 40e^{-0.4t}$. Therefore, $e^{0.32t} = 5$. Hence, $t = \frac{25}{8} \ln 5$ is a critical number. (One can use appropriate derivative tests to deduce that a local maximum (and hence a maximum) value is attained at $t = \frac{25}{8} \ln 5$.)

Grading scheme.

- (2M) Correct answer $t = \frac{25}{8} \ln 5$.

8. In this question, we demonstrate how the method of variation of parameters works on second order linear differentiation equations with *non-constant coefficients.*

(a) (2%) Verify that $e^t$ and $t+1$ satisfy the differential equation

$$ty'' - (t+1)y' + y = 0.$$

(b) (4%) Let $u_1(t), u_2(t)$ be such that $u_1'(t)\cdot e^t + u_2'(t)\cdot(t+1) = 0$. Let $y_p(t) = u_1(t)\cdot e^t + u_2(t)\cdot(t+1)$. Simplify and express

$$ty_p'' - (t+1)y_p' + y_p$$

in terms of $u_1'(t)$ and $u_2'(t)$.

(c) (6%) Find the general solution to the differential equation

$$ty'' - (t+1)y' + y = t^2.$$

**Solution:**

(a) Just verify.

(b)

$$\begin{aligned} y_p(t) &= u_1(t)\cdot e^t + u_2(t)\cdot(t+1) \\ y_p'(t) &= u_1'(t)\cdot e^t + u_2'(t)\cdot(t+1) + u_1(t)\cdot e^t + u_2(t)\cdot 1 = u_1(t)e^t + u_2(t) \\ y_p''(t) &= u_1(t)e^t + u_1'(t)e^t + u_2'(t) \\ ty_p'' - (t+1)y_p' + y_p &= t(u_1(t)e^t + u_1'(t)e^t + u_2'(t)) - (t+1)(u_1(t)e^t + u_2(t)) + u_1(t)\cdot e^t + u_2(t)\cdot(t+1) \\ &= u_1'(t)te^t + tu_2'(t) \end{aligned}$$

(c)

Solve the system

$$\begin{aligned} u_1'(t)\cdot e^t + u_2'(t)\cdot(t+1) &= 0 \\ u_1'(t)\cdot(te^t) + u_2'(t)\cdot t &= t^2 \end{aligned}$$

To get

$$\begin{aligned} u_1'(t) = (t+1)e^{-t} \quad &, \quad u_2'(t) = -1 \\ u_1(t) &= -(t+1)e^{-t} - e^{-t} + C_1 \\ u_2(t) &= -t + C_2 \end{aligned}$$

Hence

$$y(t) = C_1e^t + C_2(t+1) - t^2 - 2t - 2$$

□

Grading:

- (a) is 1% per verify.
- (-2%) for each mistake in (b) (because it will change the student's answer in (c)).
- (c) can be done with undetermined coefficients. In that case, (-2%) for each mistake.
- 1% for knowing the system of equations to solve. 1% for solving $u_1'$ and $u_2'$. 1% for integrating each. 2% for putting all the information together as a final answer.
- Note: $y(t) = C_1e^t + C_2(t+1) - t^2$ is also a solution but if students use undetermined coefficient with just $At^2$, then it is a lucky guess and not correct.