112 模組15-17班 微積分1 期考解答和評分標準

1. Evaluate.

(a) (6 pts) $\lim_{x\to 0}\frac{\tan x+\cot x}{\sin x+\csc x}$ (b) (6 pts) $\lim_{x\to\infty}\frac{3x+5}{\ln(7+9e^x)}$ (c) (6 pts) $\lim_{x\to 0^+}(1-\cos x)^{\sin x}$

Solution:

(a)

$$\lim_{x\to 0}\frac{\tan x+\cot x}{\sin x+\csc x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}{\sin x+\frac{1}{\sin x}}$$
$$=\lim_{x\to 0}\frac{\sin^2 x+\cos^2 x}{(\sin^2 x+1)\cos x}=\frac{1}{(1+0)\cdot 1}=1$$

(b) The given limit is $\frac{\infty}{\infty}$-type indeterminate form, hence we can use l'Hospital's rule.

$$\lim_{x\to\infty}\frac{3x+5}{\ln(7+9e^x)}=\lim_{x\to\infty}\frac{3}{\frac{9e^x}{7+9e^x}}$$
$$=\lim_{x\to\infty}\frac{7+9e^x}{3e^x}=\lim_{x\to\infty}\frac{7}{3}e^{-x}+3=3$$

(c) The given limit is 0^0-type indeterminate form, hence we can use l'Hospital's rule. We first let $y=(1-\cos x)^{\sin x}$, then

$$\ln y=\sin x\ln(1-\cos x)$$

Then we compute

$$\lim_{x\to 0^+}y=\lim_{x\to 0^+}\frac{\ln(1-\cos x)}{\csc x}$$
$$=\lim_{x\to 0^+}\frac{\frac{\sin x}{1-\cos x}}{-\cot x\csc x}=\lim_{x\to 0^+}\frac{-\sin^3 x}{(1-\cos x)\cos x}$$
$$=\lim_{x\to 0^+}\frac{-(1-\cos^2 x)\sin x}{(1-\cos x)\cos x}=\lim_{x\to 0^+}\frac{-(1+\cos x)\sin x}{\cos x}=0.$$

and therefore $\lim_{x\to 0^+}y=e^0=1$.

Grading criteria: Any computation mistake −1. In (b) and (c), checking the given limit is in indeterminate form +2, computing derivative correctly +2. +2 point for the right answer.

2. Differentiate.

(a) (6 pts) $y = \ln\left(\tan^{-1}(e^x)\right)$

(b) (6 pts) $f(x) = \tan\left(\dfrac{x}{1+x^2}\right)$

(c) (6 pts) $g(x) = x^{\cos x}$

Solution:

(a)

$$\frac{dy}{dx} = \frac{1}{\tan^{-1}(e^x)} \cdot \frac{1}{1+e^{2x}} \cdot e^x = \frac{e^x}{\tan^{-1}(e^x)(1+e^{2x})}$$

(b)

$$f'(x) = \sec^2\left(\frac{x}{1+x^2}\right) \cdot \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} \sec^2\left(\frac{x}{1+x^2}\right)$$

(c)

$$g'(x) = x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right)$$

□

Grading:

(a)

The derivatives of $\ln(\cdot)$, $\tan^{-1}(\cdot)$, e^x are each worth (0.5 pts).

(4 pts) for using Chain Rule correctly. Forgetting a term is (-2 pts) and forgetting to plug in inner function is (-1 pt) each.

(0.5 pts) for having an answer and not making any extra mistakes.

Examples:

$\dfrac{1}{\tan^{-1}(e^x)} \cdot \dfrac{1}{1+e^{2x}}$ will get 3/6

$\dfrac{1}{x} \cdot \dfrac{1}{1+x^2} \cdot e^x$ will get 1.5/6

(b)

The derivatives of each piece of the function are each worth (0.5 pts).

(4 pts) for using Chain Rule and Quotient Rule correctly.

(0.5 pts) for having an answer and not making any extra mistakes.

(c)

The derivatives of each piece of the function are each worth (0.5 pts).

(4 pts) for using Logarithmic Differentiation (or use change of base to e) and Product Rule correctly.

(0.5 pts) for having an answer and not making any extra mistakes.

3. (a) (8 pts) Find an equation of the tangent line to the curve at the given point.

$$y = \sqrt{x + \sqrt{x+y}}\ , \quad (0,1)$$

(b) (4 pts) The equation in (a) defines a function $y = f(x)$ near $(0,1)$. Use linear approximation to estimate $f(0.02)$.

Solution:

(a) To simplify a little, we take square on each side.

$$y^2 = x + \sqrt{x+y}.$$

Then we use implicit differentiation.

$$2yy' = 1 + \frac{1+y'}{2\sqrt{x+y}}.$$

We evaluate this equation at $(0,1)$.

$$2y' = 1 + \frac{1+y'}{2\sqrt{0+1}} = \frac{3}{2} + \frac{y'}{2}. \tag{1}$$

Solving the above equation for y', we obtain $y' = 1$. Therefore, the equation of the tangent line at $(0,1)$ is

$$y = x + 1.$$

(b) Since 0.02 is very close to 0, we can approximate the y value at $x = 0.02$ with the tangent line of the curve.

$$y|_{x=0.02} \approx 0.02 + 1 = 1.02$$

Grading criteria: Any computation mistake −1. In part (a), Taking implicit differentiation correctly +3. Evaluating correctly +2. Correct y' +1, and correct equation of the tangent line +2. In part (b), Evaluating the tangent line from (a) +3, correct answer +1.

4. (18 pts) True or false. For each statement below, write T if the statement is true and F if the statement is false.

(a) **T** The function $f(x) = |x|$ is not differentiable at $x = 0$.

(b) **F** The function $f(x) = |x^3|$ is not differentiable at $x = 0$.

(c) **F** If $f(1) = 10$ and $f'(x) \geq 2$ for all $x \in \mathbb{R}$, then $f(4)$ cannot be larger than 16.

(d) **T** If $f(1) = 10$ and $f'(x) \geq 2$ for all $x \in \mathbb{R}$, then $f(-4)$ cannot be positive.

(e) **F** The function $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is a continuous function.

(f) **T** The function $f(x) = \begin{cases} \frac{1 - \cos x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is a continuous function.

Solution:

(a) **T**

Use the definition of differentiation, the left limit at $x = 0$ equals to -1, while the right limit at $x = 0$ equals to 1.

(b) **F**

Use the definition of differentiation, both the left and right limits at $x = 0$ are equal to 0.

(c) **F** Apply the mean value theorem

$$\frac{f(4) - f(1)}{4 - 1} = f'(c) \geqslant 2 \Rightarrow f(4) \geqslant 16.$$

(d) **T** Apply mean value theorem

$$\frac{f(1) - f(-4)}{1 - (-4)} = f'(c) \geqslant 2 \Rightarrow f(-4) \leqslant 0$$

(e) **F** The limit of $f(x)$ at $x = 0$ is equal to 1, while $f(0) = 0$.

(f) **T** Both the limit of $f(x)$ at $x = 0$ and $f(0)$ are equal to 0.

(**Each correct answer worth 3pts.**)

5. Consider $f(x) = \frac{1}{x} + \ln x$ over the closed interval $\left[\frac{1}{2}, 4\right]$.

(a) (6 pts) Compute $f'(x)$. Find the intervals of increase or decrease.

(b) (6 pts) Compute $f''(x)$. Find the intervals of concavity.

(c) (4 pts) Given that $\ln 2 \approx 0.7$. Sketch the graph $y = f(x)$. Label any local maximum, local minimum, or inflection point.

(d) (4 pts) Find the absolute maximum and minimum value of $f(x)$ over $\left[\frac{1}{2}, 4\right]$.

Solution:

(a)

$$f'(x) = \frac{-1}{x^2} + \frac{1}{x} = \frac{x-1}{x^2}$$

Interval of increase: $(1, \infty)$

Interval of decrease: $(0, 1)$

(b)

$$f''(x) = \frac{2}{x^3} + \frac{-1}{x^2} = \frac{2-x}{x^3}$$

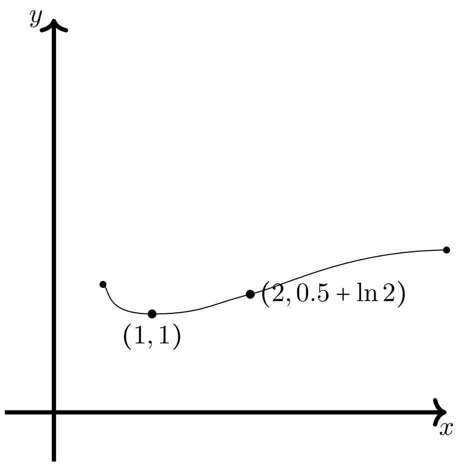
Concave up interval: $(0, 2)$

Concave down interval: $(2, \infty)$

(c)

The points to label: $(1,1)$ local minimum, $(2, 0.5 + \ln 2) \approx (2, 1.2)$ inflection point.

The endpoints $(0.5, 2 - \ln 2)$, $(4, 0.25 + 2\ln 2)$



(d)

The absolute maximum occurs at $x = 4$, with value $f(4) \approx 1.65$

The absolute minimum occurs at $x = 1$, with value $f(1) = 1$

□

Grading:

In this problem students can get points for later parts even if they make a mistake early.

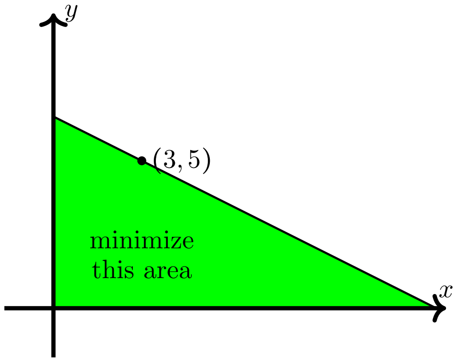
(a) Derivative is (2 pts) and determining the intervals (4 pts).

(b) Second derivative is (2 pts) and determining the intervals (4 pts).

(c) Sketch is (2 pts), the points labeling is (2 pts).

(d) Finding candidates (critical number) is (2 pts). Verifying the points for max and min is (2 pt).

6. (14 pts) What is the equation of the line through the point $(3, 5)$ that cuts off the least area from the first quadrant (see figure)? Also find the minimum area.



Solution:

Assume the line pass through $(3, 5)$ has x-intercept $(a, 0)$, where $a > 0$. Then the equation of the line is

$$L : y - 5 = \frac{5}{3 - a}(x - 3).$$

(**Writing down the general line equation gets 1pt**)

Its y-intercept is $(0, \frac{15}{a-3} + 5)$. Hence the area cut off by the line and the first quadrant makes sense only when $a > 3$. And the area is parametrized by the following equation of a:

$$A(a) = \frac{5a^2}{2(a - 3)}, \text{ where } a > 3.$$

(**Writing down the Area function get 5pts**)

Our goal is to find the minimum of $A(a)$ when $a > 3$. Firstly,

$$A'(a) = \frac{10a^2 - 60a}{4(a - 3)^2},$$

the only critical number when $a > 3$ is $a = 6$.

(**Correctly find out critical numbers get 3pts**)

Moreover,

$$A''(a) = \frac{45}{(a - 3)^3}.$$

Thus $A''(6) > 0$, and so $a = 6$ is the minimum point of $A(a)$ when $a > 3$.

(**Correctly test the critical number is local minimum get 3pts**)

Therefore, the desired line having minimal cut-off area from the first quadrant is

$$L_{a=6} : y - 5 = \frac{-5}{3}(x - 3).$$

And the minimal area is

$$A(6) = 30.$$

(**Final answer worth 2pts**)