1. (20%) Evaluate the following limits.

   (a) (5%) $\lim_{x\to 0} |\sin x| \sin(\frac{2}{x})$.

   (b) (5%) $\lim_{t\to 0} \frac{1}{t\sqrt{1+t}} - \frac{1}{t}$

   (c) (5%) $\lim_{x\to\infty} \frac{x^2+2e^{-x}}{e^{-x}+3\,e^{x}}$, $\lim_{x\to-\infty} \frac{x^2+2e^{-x}}{e^{-x}+3\,e^{x}}$.

   (d) (5%) $\lim_{x\to 1^+} x^{1/(x-1)}$.

Solution:

(a) Note that $-1 \le \sin(\frac{2}{x}) \le 1$. Hence

$$-|\sin x| \le |\sin x| \cdot \sin(\frac{2}{x}) \le |\sin x| \qquad \text{for all } x.$$

Moreover, $\lim_{x\to 0} -|\sin x| = 0 = \lim_{x\to 0} |\sin x|$. Therefore, by the squeeze theorem, $\lim_{x\to 0} |\sin x| \sin(\frac{2}{x}) = 0$.

(1 pt for using the squeeze theorem. 2 pts for the correct inequality $-|\sin x| \le |\sin x| \cdot \sin(\frac{2}{x}) \le |\sin x|$. 1 pt for checking the limits of $-|\sin x|$ and $|\sin x|$. 1 pt for the final answer.)

(b) Solution 1:

$$\lim_{t\to 0} \frac{1}{t\sqrt{1+t}} - \frac{1}{t} = \lim_{t\to 0} \frac{1-\sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t\to 0} \frac{(1-\sqrt{1+t})(1+\sqrt{1+t})}{t\sqrt{1+t}\cdot(1+\sqrt{1+t})}$$
$$= \lim_{t\to 0} \frac{-t}{t\sqrt{1+t}\cdot(1+\sqrt{1+t})} = \lim_{t\to 0} \frac{-1}{\sqrt{1+t}\cdot(1+\sqrt{1+t})} = -\frac{1}{2}.$$

(1 pt for reduction of fractions to a common denominator. 3 pts for rationalizing the numerator and simplifying the quotient. 1 pt for the final answer.)

Solution 2:

$$\lim_{t\to 0} \frac{1}{t\sqrt{1+t}} - \frac{1}{t} = \lim_{t\to 0} \frac{1-\sqrt{1+t}}{t\sqrt{1+t}} \overset{\frac{0}{0} L'H}{=} \lim_{t\to 0} \frac{-\frac{1}{2\sqrt{1+t}}}{\sqrt{1+t}+\frac{t}{2\sqrt{1+t}}} = -\frac{1}{2}.$$

(1 pt for reduction of fractions to a common denominator. 3 pts for correctly applying l'Hospital's Rule. 1 pt for the final answer.)

(c) Solution 1:

$$\lim_{x\to\infty} \frac{x^2+2e^{-x}}{e^{-x}+3e^{x}} = \lim_{x\to\infty} \frac{\frac{x^2}{e^x}+2e^{-2x}}{e^{-2x}+3}.$$

Since $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$ (by l'Hospital's Rule) and $\lim_{x\to\infty} e^{-2x} = 0$, we have

$$\lim_{x\to\infty} \frac{x^2+2e^{-x}}{e^{-x}+3e^{x}} = \lim_{x\to\infty} \frac{\frac{x^2}{e^x}+2e^{-2x}}{e^{-2x}+3} = \frac{0+0}{0+3} = 0.$$

(0.5 pt for dividing the dominate term e^x. 1 pt for knowing that $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$ and $\lim_{x\to\infty} e^{-2x} = 0$. 0.5 pt for the final answer.)

$$\lim_{x\to-\infty} \frac{x^2+2e^{-x}}{e^{-x}+3e^{x}} = \lim_{x\to-\infty} \frac{x^2e^x+2}{1+3e^{2x}}.$$

Since $\lim_{x\to-\infty} x^2e^x = \lim_{x\to-\infty} \frac{x^2}{e^{-x}} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to-\infty} \frac{2x}{-e^{-x}} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0$ and $\lim_{x\to-\infty} e^{2x} = 0$, we have

$$\lim_{x\to-\infty} \frac{x^2+2e^{-x}}{e^{-x}+3e^x} = \lim_{x\to-\infty} \frac{x^2e^x+2}{1+3e^{2x}} = \frac{0+2}{1+0} = 2.$$

(1 pt for dividing the dominate term e^{-x}. 1 pt for $\lim_{x\to-\infty} x^2e^x = 0$. 1 pt for the final answer.)

Solution 2

$$\lim_{x\to\infty} \frac{x^2+2e^{-x}}{e^{-x}+3e^x} \overset{\frac{\infty}{\infty} L'H}{=} \lim_{x\to\infty} \frac{2x-2e^{-x}}{-e^{-x}+3e^x} \overset{\frac{\infty}{\infty} L'H}{=} \lim_{x\to\infty} \frac{2+2e^{-x}}{e^{-x}+3e^x}.$$

Because the numerator tends to 2 and the denominator tends to infinity, the above limit is 0.

(1 pt gor applying l'Hospital's Rule correctly. 1 pt for the final answer.)

$$\lim_{x\to-\infty} \frac{x^2+2e^{-x}}{e^{-x}+3e^x} \overset{\frac{\infty}{\infty} L'H}{=} \lim_{x\to-\infty} \frac{2x-2e^{-x}}{-e^{-x}+3e^x} \overset{\frac{\infty}{\infty} L'H}{=} \lim_{x\to-\infty} \frac{2+2e^{-x}}{e^{-x}+3e^x} = \lim_{x\to-\infty} \frac{2e^x+2}{1+3e^{2x}}.$$

Moreover, $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to-\infty} e^{2x} = 0$. Hence,

$$\lim_{x\to-\infty} \frac{x^2+2e^{-x}}{e^{-x}+3e^x} = \lim_{x\to-\infty} \frac{2e^x+2}{1+3e^{2x}} = \frac{0+2}{1+0} = 2.$$

(1 pt gor applying l'Hospital's Rule correctly. 1 pt for $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to-\infty} e^{2x} = 0$. 1 pt for the final answer.)

(d) $\ln[x^{1/(x-1)}] = \frac{\ln x}{x-1}$. Moreover, $\lim_{x\to1^+} \frac{\ln x}{x-1} \overset{\frac{0}{0} L'H}{=} \lim_{x\to1^+} \frac{\frac{1}{x}}{1} = 1$.
Hence $\lim_{x\to1^+} x^{1/(x-1)} = e^1 = e$.

(1 pt for taking natural logarithm. 3 pts for computing the limit $\lim_{x\to1^+} \frac{\ln x}{x-1} = 1$. 1 pt for the final answer e.)

If, during the limit calculation, students first evaluate part of the limit, such as considering that e^{-x} goes to 0 as x goes to infinity in the denominator or numerator, and then evaluate the entire limit, they will be deducted 3 points.

2. (15%) Find the following derivatives.

(a) (5%) $f(x) = \tan^{-1}(\sqrt{x})$. Find $f'(x)$.

(b) (5%) Given $x^4 - 4x^2 + 2y^4 + xy = 0$, find $\frac{dy}{dx}$ at $(x, y) = (1, 1)$.

(c) (5%) $f(x) = x^{\sin x}$. Find $f'(x)$.

Solution:

(a) $f(x) = \tan^{-1}\sqrt{x}$. Find $f'(x)$.

Let $y = \sqrt{x}$, then $f(x) = \tan^{-1}(y)$ (2%).

$$f'(x) = \frac{1}{1+y^2} \cdot \frac{1}{2\sqrt{x}}. \quad (4\%)$$

$$= \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}}. \quad (5\%)$$

(b) Given $x^4 - 4x^2 + 2y^4 + xy = 0$. Find $\frac{dy}{dx}$ at $(x, y) = (1, 1)$.

$$4x^3 - 8x + 8y^3 \cdot y' + y + x \cdot y' = 0. \quad (3\%)$$

$$(8y^3 + x)y' + (4x^3 - 8x + y) = 0.$$

At $(x, y) = (1, 1)$, $y' = \frac{1}{3}$. (+2%).

(c) $f(x) = x^{\sin x}$. Find $f'(x)$.

Consider

$$\ln f(x) = \sin x \cdot \ln x \quad (2\%)$$

$$\frac{f'(x)}{f(x)} = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x} \quad (+2\%)$$

$$f'(x) = (\cos x \cdot \ln x + \sin x \cdot \frac{1}{x}) \cdot x^{\sin x} (+1\%)$$

3. (a) (5%) It is known that $\cos(x)\cdot\cos(2x) = \dfrac{\sin(4x)}{4\sin(x)}$ for every real number x which is not an integer multiple of π. Use this identity or other methods to show that

$$\tan(x) + 2\tan(2x) = \cot(x) - 4\cot(4x)$$

for every real number x which is not an integer multiple of $\pi/4$. (Hint: Use the logarithmic differentiation to differentiate the known identity.)

(b) (5%) Evaluate $\displaystyle\lim_{x\to 0}\frac{\cot(x) - 4\cot(4x)}{x}$.

Solution:

(a) **(Method 1)** Taking $\ln|\cdot|$ on both sides of the known identity, we get

$$\ln|\cos(x)| + \ln|\cos(2x)| = \ln|\sin(4x)| - \ln 4 - \ln|\sin(x)|. \text{ (1\%)}$$

Differentiating this identity with respect to x, we obtain

$$\frac{-\sin(x)}{\cos(x)} + \frac{-2\sin(2x)}{\cos(2x)} = \frac{4\cos(4x)}{\sin(4x)} - \frac{\cos(x)}{\sin(x)}, \text{ (3\%)}$$

which implies immediately $\tan(x) + 2\tan(2x) = \cot(x) - 4\cot(4x)$. (1%) □

(Method 2) Differentiating the known identity directly, we get

$$-\sin(x)\cos(2x) + \cos(x)\cdot(-2\sin(2x)) = \frac{1}{4}\cdot\frac{4\cos(4x)\sin(x) - \sin(4x)\cos(x)}{\sin^2(x)}, \text{ (3\%)}$$

which, after being divided by $-\cos(x)\cos(2x)$ on the left hand side and by $-\dfrac{\sin(4x)}{4\sin(x)}$ on the right hand side (notice that $-\cos(x)\cos(2x) = -\dfrac{\sin(4x)}{4\sin(x)}$ by the known identity) (1%), yields the desired identity $\tan(x) + 2\tan(2x) = \cot(x) - 4\cot(4x)$. (1%) □

(Method 3) Using the identity $\tan(2x) = \dfrac{2\tan(x)}{1 - \tan^2(x)}$ (1%) and letting $t = \tan(x)$, we get

$$\tan(4x) = \frac{2\tan(2x)}{1 - \tan^2(2x)} = \frac{2\cdot\frac{2t}{1-t^2}}{1 - (\frac{2t}{1-t^2})^2} = \frac{4t - 4t^3}{t^4 - 6t^2 + 1} \text{ (1\%)},$$

so that

$$\begin{aligned}\tan(x) + 2\tan(2x) - \cot(x) + 4\cot(4x) &= t + \frac{4t}{1-t^2} - \frac{1}{t} + \frac{t^4 - 6t^2 + 1}{t - t^3}\\ &= \frac{(t^2 - t^4) + 4t^2 - (1 - t^2) + (t^4 - 6t^2 + 1)}{t - t^3} = 0, \text{ (2\%)}\end{aligned}$$

whence the desired identity $\tan(x) + 2\tan(2x) = \cot(x) - 4\cot(4x)$. (1%) □

(b) **(Method 1)** Dividing the shown identity of (a) by x, we get

$$\frac{\tan(x)}{x} + 2\cdot\frac{\tan(2x)}{x} = \frac{\cot(x) - 4\cot(4x)}{x}. \text{ (1\%)}$$

As $x \to 0$, we have $\dfrac{\tan(x)}{x} \to 1$ and $\dfrac{\tan(2x)}{x} \to 2$ (by l'Hospital's rule for type 0/0, or by definition of the derivative of $\tan(x)$ at $x = 0$, or by $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and $\displaystyle\lim_{x\to 0}\frac{\sin(x)}{x} = 1$) (3% here: 1% for the answers, and 2% for the proofs), so $\displaystyle\lim_{x\to 0}\frac{\cot(x) - 4\cot(4x)}{x} = \lim_{x\to 0}\left(\frac{\tan(x)}{x} + 2\cdot\frac{\tan(2x)}{x}\right) = 1 + 2\cdot 2 = 5$. (1%) □

(Method 2) Without using (a), we may write

$$\frac{\cot(x) - 4\cot(4x)}{x} = \frac{\cos(x)\sin(4x) - 4\cos(4x)\sin(x)}{x\sin(x)\sin(4x)} = \frac{g(x)}{f(x)}, \text{ (1\%)}$$

where $f(x) = x\sin(x)\sin(4x)$ and $g(x) = \cos(x)\sin(4x) - 4\cos(4x)\sin(x)$, and then use l'Hospital's rule to evaluate $\lim_{x\to 0}\frac{g(x)}{f(x)}$; in this way, one will find that $\lim_{x\to 0}\frac{g(x)}{f(x)}$, $\lim_{x\to 0}\frac{g'(x)}{f'(x)}$ and $\lim_{x\to 0}\frac{g''(x)}{f''(x)}$ are all of type 0/0, so one will need to use l'Hospital's rule three times (1% for each correct use of l'Hospital's rule; 3% in total here) to find that $\lim_{x\to 0}\frac{\cot(x) - 4\cot(4x)}{x} = \lim_{x\to 0}\frac{g'''(x)}{f'''(x)} = \frac{g'''(0)}{f'''(0)} = 5$. (1%) □

(Method 3) Again without using (a), we write

$$\frac{\cot(x) - 4\cot(4x)}{x} = \frac{\cos(x)\sin(4x) - 4\cos(4x)\sin(x)}{x\sin(x)\sin(4x)} = \frac{g(x)}{4x^3}\cdot\frac{x}{\sin(x)}\cdot\frac{4x}{\sin(4x)}, \text{ (1\%)}$$

where $g(x) = \cos(x)\sin(4x) - 4\cos(4x)\sin(x)$ is as in Method 2. Then, as in Method 2, we apply l'Hospital's rule three times to find that $\lim_{x\to 0}\frac{g(x)}{4x^3} = \lim_{x\to 0}\frac{g'''(x)}{(4x^3)'''} = \frac{g'''(0)}{24} = 5$ (1% for each correct use of l'Hospital's rule; 3% in total here), from which we conclude that $\lim_{x\to 0}\frac{\cot(x) - 4\cot(4x)}{x} = 5\cdot 1\cdot 1 = 5$ (using also $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$). (1%) □

4. (12%) Let $f(x) = 2x^2 + \ln(x)$ for $x > 0$. It is known that the function f is strictly increasing and admits an inverse function f^{-1}, such that for $x > 0$ and $y \in \mathbb{R}$ we have $y = f(x)$ if and only if $x = f^{-1}(y)$.

(a) (5%) Calculate the derivatives $f'(1)$ and $(f^{-1})'(2)$. (Observe that $f(1) = 2$.)

(b) (2%) Estimate the value $f^{-1}(2.06)$ by the linear approximation of $f^{-1}(y)$ at $y = 2$.

(c) (5%) Show that $2x^2 \leq f(x) \leq 2x^2 + x$ for all $x \geq 1$.

Solution:

(a)
- The calculation of $f'(1)$ is worth 2%: We have $f'(x) = 4x + \dfrac{1}{x}$ (1%), so $f'(1) = 5$. (1%) ☐
- The calculation of $(f^{-1})'(2)$ is worth 3%:

  (Method 1) $(f^{-1})'(2) = (f^{-1})'(f(1)) \overset{(2\%)}{=} 1/f'(1) \overset{(1\%)}{=} 1/5$. ☐

  (Method 2) From $f(x) = 2x^2 + \ln(x)$, we have $y = 2(f^{-1}(y))^2 + \ln(f^{-1}(y))$ (1%) which, after being differentiated with respect to y, yields $1 = 4f^{-1}(y) \cdot (f^{-1})'(y) + \dfrac{(f^{-1})'(y)}{f^{-1}(y)}$ (1%); setting $y = 2$ therein and using $f^{-1}(2) = 1$, we get $1 = 4 \cdot 1 \cdot (f^{-1})'(2) + \dfrac{(f^{-1})'(2)}{1}$ and hence $(f^{-1})'(2) = 1/5$. (1%) ☐

(b) By the linear approximation of $f^{-1}(y)$ at $y = 2$, for small $|t|$ we have

$$f^{-1}(2+t) \approx f^{-1}(2) + (f^{-1})'(2) \cdot t = 1 + (t/5), \text{ (1\%)}$$

so (setting $t = 0.06$) $f^{-1}(2.06) \approx 1 + (0.06/5) = 1.012$. (1%)

Note that the linear approximation can of course be written as $f^{-1}(y) \approx f^{-1}(2) + (f^{-1})'(2) \cdot (y-2)$ for y near 2, in which case one will then set $y = 2.06$ therein. ☐

(c)
- The inequality $2x^2 \leq f(x)$ is worth 1%: For $x \geq 1$, we have $\ln x \geq \ln 1 = 0$ ($x \mapsto \ln x$ is an increasing function), so that $f(x) \geq 2x^2$. (1%) ☐
- The inequality $f(x) \leq 2x^2 + x$ is worth 4%:

  (Method 1) Consider $g(x) = x - \ln(x)$. We have $g(1) = 1$ and $g'(x) = 1 - \dfrac{1}{x} \geq 0$ for $x \geq 1$ (1%), so $g(x)$ is increasing on the interval $x \geq 1$ (1%) and hence $g(x) \geq g(1) = 1 > 0$ for all $x \geq 1$ (1%). This shows that for $x \geq 1$ we have $\ln(x) \leq x$ and hence $f(x) \leq 2x^2 + x$. (1%) ☐

  (Method 2) By the mean value theorem, for every $x > 1$ there is a real number $t \in (1, x)$ such that

$$\ln(x) = \ln(x) - \ln(1) \overset{(1\%)}{=} \ln'(t) \cdot (x-1) \overset{(1\%)}{=} \frac{x-1}{t} \overset{(1\%)}{\leq} x - 1 < x.$$

  But we also have $\ln(1) = 0 < 1$. Thus for $x \geq 1$ we have $\ln(x) < x$ and hence $f(x) \leq 2x^2 + x$. (1%) ☐

5. (22%) Consider the function $f(x) = \dfrac{(x-1)^3}{(x+1)^2} = x - 5 + \dfrac{12x+4}{(x+1)^2}$ for $x \neq -1$.

(a) (7%) Find $f'(x)$. Write down the interval(s) of increase and interval(s) of decrease of $f(x)$.

(b) (7%) Find $f''(x)$. Write down the interval(s) on which $f(x)$ is concave upward and the interval(s) on which $f(x)$ is concave downward.

(c) (2%) Write down (if any) the local extremas and inflection points.

(d) (3%) Find all the asymptotes of $y = f(x)$.

(e) (3%) Sketch the graph of $y = f(x)$.

Solution:

(a) (2M) $f'(x) = \dfrac{3(x-1)^2(x+1) - (x-1)^3 \cdot 2(x+1)}{(x+1)^4} = \dfrac{(x-1)^2(x+5)}{(x+1)^3}$.
The critical numbers are $x = 1$, $x = -5$.
(1M) $f'(x) > 0$ for $x > -1$ or $x < -5$; $f'(x) < 0$ for $-5 < x < -1$.
(4M) Interval of increase $(-\infty, -5) \cup (-1, \infty)$; Interval of decrease $(-5, -1)$.

Marking scheme for 5a
2M - correct $f'(x)$ (1M for knowing the quotient rule)
1M - for knowing $f'(x) > 0$ (resp. < 0) implies f is increasing (resp. decreasing)
4M - identify correctly the monotoncity on each of the following intervals : $(-\infty, -5)$, $(-5, -1)$, $(-1, 1)$ $(1, \infty)$.

(b) Since $f'(x) = (x-1)^2(x+5)(x+1)^{-3}$, by product rule,

$$(3\text{M}) f''(x) = 2(x-1)(x+5)(x+1)^{-3} + (x-1)^2(x+1)^{-3} + (x-1)^2(x+5)(-3(x+1)^{-4}) = \frac{24(x-1)}{(x+1)^4}.$$

(1M) $f''(x) > 0$ when $x > 1$; $f''(x) < 0$ when $x < -1$ or $-1 < x < 1$
(3M) Concave upward : $(1, \infty)$; Concave downward : $(-\infty, -1) \cup (-1, 1)$

Marking scheme for 5b
3M - correct $f''(x)$ (partial credits available)
1M - for knowing $f''(x) > 0$ (resp. < 0) implies f is concave upward (resp. downward)
3M - identify correctly the monotoncity on each of the following intervals : $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$.
(-0.5M for those who did not remove $x = -1$ from the interval)

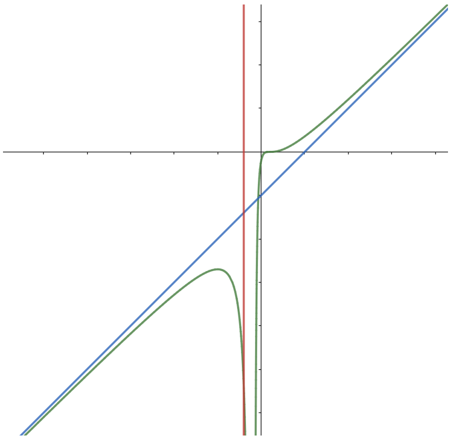
(c) (1M) Local maximum : $(-5, f(-5) = -\dfrac{27}{2})$
Local minimum : NONE
(1M) Inflection point : $(1, f(1) = 0)$

Marking scheme for 5c
0.5M+0.5M - correct local max. (each coordinate)
0.5M+0.5M - correct inflection points (each coordinate)

(d) (0.5M) $x = -1$ is a vertical asymptote,
(0.5M) because $\lim_{x \to -2} f(x) = -\infty$
Let $y = ax + b$ be a slant asymptote (towards ∞).
(0.5M) $a = \lim_{x \to \infty} \dfrac{(x-1)^3}{x(x+1)^2} = 1$.
(0.5M) $b = \lim_{x \to \infty} \dfrac{(x-1)^3}{(x+1)^2} - x = -5$.
(0.5M) So $y = x - 5$ is a slant asymptote (towards ∞).
(0.5M) The calculation for $x \to -\infty$ is identical.
Hence $y = x - 5$ is the slant asymptote.

Marking scheme for 5d
0.5M - correct vertical asymptote
0.5M - correct verification of the vertical asymptote
0.5M - computation of a
0.5M - computation of b
0.5M - correct slant asymptote
(We accept students writing '$y = x - 5$ is a slant asymptote towards $+\infty$ because $\lim_{x\to\infty}(f(x) - x + 5) = \lim_{x\to\infty} \frac{12x+4}{(x+1)^2} = 0$'. In this case, 0.5M for correct asymptote and 1M on verifying the relevant limit.)
0.5M - awareness of analysing the slant asymptote at $-\infty$

(e) Sketch :



Marking scheme for 5e
(0.5M) Asymptotes $x = -1$ and $y = x - 5$
(0.5M) Local max at $(-5, -13.5)$
(0.5M) Inflection point at $(1, 0)$
(0.5M)The shape for $x < -1$
(0.5M) The shape for $-1 < x < 1$
(0.5M) The shape for $x > 1$

6. (9%) The morning shift in a certain factory, which lasts from 8:00 A.M. to 12:15 P.M., consists of 4 hours of working time and a 15-minute coffee break. An average worker who starts to work at 8:00 A.M. in this factory assembles $f(t) = -t^3 + 6t^2 + 15t$ units in t hours before the coffee break; after the 15-minute coffee break, he can assemble $g(t) = -\frac{1}{3}t^3 + t^2 + 23t$ units in t hours.

(a) (3%) If an average worker in this factory works for t hours before the coffee break, $0 \le t \le 4$, find the total units he can assemble during the morning shift. (Hint: There are $4 - t$ working hours after the coffee break.)

(b) (6%) Find the optimal number of hours an average worker in this factory should work before the coffee break to maximize the number of units assembled by 12:15 P.M.. (You need to justify that the answer you find is indeed the absolute maximum.)

Solution:

(a) The total units a worker can assemble during the morning shift is

$$h(t) = f(t) + g(4-t) = -t^3 + 6t^2 + 15t + (-\frac{1}{3})(4-t)^3 + (4-t)^2 + 23(4-t).$$

(1 pt for knowing that the total units is $f(t) + g(4-t)$. 2 pts for the final answer.)

(b) Find the maximum value of $h(t)$ for $0 \le t \le 4$.

$$h'(t) = -3t^2 + 12t + 15 + (4-t)^2 - 2(4-t) - 23 = -2t^2 + 6t = -2t(t-3).$$

$h'(t) = 0$ if $t = 0$ or $t = 3$. Moreover, $h'(t) > 0$ for $0 < t < 3$ and $h'(t) < 0$ for $3 < t < 4$. Hence the absolute maximum value of $h(t)$ on $[0, 4]$ occurs when $t = 3$.

Or we can compare the values of $h(3), h(0), h(4)$. Since $h(3) = 95\frac{2}{3}$, $h(0) = 86\frac{2}{3}$, $h(4) = 92$, and $h(3) > h(4) > h(0)$, we conclude that $h(3)$ is the absolute maximum value of $h(t)$ on $[0, 4]$.

(2 pts for computing $h'(t)$. 2 pts for finding critical numbers of $h(t)$. 2 pts for verifying that $h(3)$ is the absolute maximum. If students only use the second derivative test to verify a local maximum, they will only receive one point.)

7. (12%) A new Pokemon Center is going to launch in Decemeber 2023 in Xinyi Distict. Let p (in thousand NTD) be the currency of Taiwan and q (in thousand yen) be the currency of Japan. The exchange rate of their currency is given by $q = 5p$. For the latest version of Snorlax doll from Pokemon Center, it is known that

- $Y(p)$ is the quantity demanded for this product when it is priced at p NTD.
- $y(q)$ is the quantity demanded for this product when it is priced at q yen.

Suppose it is known that $Y(p) = 100 + \dfrac{100}{\sqrt{p}} - \dfrac{p}{10}$ for $1 \leq p \leq 100$. Let the point elasticity of demand of the Snorlax doll be $\varepsilon(p) = \dfrac{Y'(p) \cdot p}{Y(p)}$.

(a) (8%) Find $\varepsilon(p)$ when $p = 16$. Should the retailer increase or decrease the price in order to obtain a higher revenue $p \cdot Y(p)$ at this price?

(b) (1%) Assume that the demand for Snorlax dolls is solely determined by their monetary value. Which of the following equality correctly captures the relation of the functions Y and y? (Circle the correct answer)

(A) $Y(5p) = y(p)$ (B) $5Y(p) = y(p)$ (C) $Y(p) = y(5p)$ (D) $Y(p) = 5y(p)$

(c) (3%) Find and prove a relation between point elasticities of demand $\varepsilon(p)$ in NTD and $\varepsilon(q) = \dfrac{y'(q) \cdot q}{y(q)}$ in Japanese yen.

Solution:

(a) (1M) As $Y'(p) = -50p^{-3/2} - \dfrac{1}{10}$,

(1M) $Y'(16) = -\dfrac{50}{64} - \dfrac{1}{10} = -\dfrac{141}{160} = -0.88125$.

(1M) Moreover $Y(16) = \dfrac{1234}{10} = 123.4$,

(1M) we have $\varepsilon(16) = \dfrac{-\frac{141}{160} \cdot 16}{\frac{1234}{10}} = -\dfrac{141}{1234} < -1$.

(2M) Since the marginal revenue equals

$$R'(p) = Y(p) + pY'(p) = Y(p)(1 + \varepsilon(p)),$$

(1M) in particular we have $R'(16) > 0$.
Therefore, the revenue is increasing at $p = 16$.
(1M) The retailer should increase the price to obtain a higher revenue.

Marking Scheme .
1M+1M+1M+1M for the correct $Y'(p)$, $Y'(16)$, $Y(16)$, $\varepsilon(16)$.
2M for relating marginal revenue $R'(p)$ and point elasticity of demand $\varepsilon(p)$
1M+1M for pointing out $R'(16) > 0$ and for the correct conclusion. (No marks will be given to the conclusion if no valid justifications are offered.)

(b) Due to the given exchange rate, we have $Y(p) = y(5p)$ (i.e. the quantity demanded when the product is priced at p thousand NTD or at $5p$ thousand yen would be the same because 'p NTD' and '$5p$' yen have the same monetary value.) The answer is C.

Marking Scheme .
All or nothing.

(c) (1M) By the chain rule $Y'(p) = 5y'(5p)$. Therefore,

$$\varepsilon(p) = \frac{Y'(p) \cdot p}{Y(p)} = \underbrace{\frac{5y'(5p) \cdot \frac{q}{5}}{y(5p)}}_{(1M)} = \frac{y'(5p) \cdot q}{y(5p)} = \frac{y'(q) \cdot q}{y(q)} \text{ as } q = 5p \underbrace{= \varepsilon(q)}_{(1M)}.$$

Marking Scheme .
1M for relating Y' and y' correctly.
1M for transforming $\varepsilon(p)$ in terms of y and y'
1M for claiming $\varepsilon(p) = \varepsilon(q)$.