1. Compute y'.

   (a) (5 pts) $y = \dfrac{e^{-2x}(1-x^2)^x}{\sqrt[3]{1+x}}$, $0 < x < 1$.

   (b) (5 pts) $y = \ln|\sec(2x) - \tan(2x)| + \tan^{-1}\left(\dfrac{1}{x}\right) + \cos^{-1}\left(\dfrac{1}{\sqrt{x^2+1}}\right)$, $x > 0$.

Solution:

(a) {Solution 1}

(STEP 1) Take natural logarithm on the both sides(1%) and correctly express the right hand side in terms of sum(1%):

$$\ln(y) = -2x + x\ln(1-x^2) - \frac{1}{3}\ln(1+x).$$

(STEP 2) Apply implicit differentiation(1%) and correctly carry out computation(1%):

$$\frac{1}{y}y' = -2 + \ln(1-x^2) + \frac{-2x^2}{1-x^2} - \frac{1}{3(1+x)}.$$

(STEP 3) Solve for y':

$$y' = \frac{e^{-2x}(1-x^2)^x}{\sqrt[3]{1+x}}\left(-2x + \ln(1-x^2) + \frac{-2x^2}{1-x^2} - \frac{1}{3(1+x)}\right).$$

{Solution 2} (STEP 1) Apply the quotient rule directly and see the multiplication rules (1%) while taking differentiation to the numerator, and correctly carry out computation (1%):

$$y' = \frac{\left(-2e^{-2x}(1-x^2)^x + e^{2x}\frac{d}{dx}\left(1-x^2\right)^x\right)(1+x)^{1/3} - \left(e^{-2x}(1-x^2)^x\right)\cdot\frac{1}{3}(1+x)^{-2/3}}{(1+x)^{2/3}}.$$

(STEP 2) Apply the logarithm differentiation to the term $(1-x^2)^x$ (1%), and correctly carry out computation (1%):

$$\ln(1-x^2) + \frac{-2x^2}{1-x^2}.$$

(STEP 3) Correctly solve y' (1%):

$$y' = \frac{\left(-2e^{-2x}(1-x^2)^x + e^{2x}(1-x^2)^x(\ln(1-x^2) - \frac{2x^2}{1-x^2})\right)(1+x)^{1/3} - \left(e^{-2x}(1-x^2)^x\right)\cdot\frac{1}{3}(1+x)^{-2/3}}{(1+x)^{2/3}}.$$

(b) (2%) (Any calculate mistake -0.5 pt, and allow no simplified answer):

$$\frac{d}{dx}(\ln|\sec(2x) - \tan(2x)|) = \frac{2\sec(2x)\tan(2x) - 2\sec^2(2x)}{\sec(2x) - \tan(2x)} = -2\sec(2x).$$

(1%) (Any calculate mistake -0.5 pt, and allow no simplified answer):

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{1}{x}\right)\right) = \frac{-\frac{1}{x^2}}{1+\left(\frac{1}{x}\right)^2} = \frac{-1}{x^2+1}.$$

(2%) (Any calculate mistake -0.5 pt, and allow no simplified answer):

$$\frac{-1}{\sqrt{(1-\left(\frac{1}{\sqrt{x^2+1}}\right)^2}}\cdot\frac{-1}{2}(x^2+1)^{-\frac{3}{2}}2x = \frac{1}{1+x^2}$$

2. Near the point $(-1, 0)$, the equation $2x^2 + 6xy + 5y^2 = 2$ defines y as an implicit function of x which is denoted by $y = f(x)$.

   (a) (4 pts) Evaluate $f'(-1)$.

   (b) (4 pts) Evaluate $f''(-1)$.

   (c) (2 pts) Use linear approximation to estimate $f(-1.03)$.

   (d) (2 pts) Is the estimate greater than or less than the actual value $f(-1.03)$? Explain.

Solution:

(a) Differentiate the equation $2x^2 + 6xy(x) + 5(y(x))^2 = 2$ with respect to x. We obtain

$$4x + 6y + 6xy' + 10yy' = 0. \qquad \text{(2 pts)}$$

At $(x, y) = (-1, 0)$, the above equation becomes $-4 - 6y' = 0$(1 pt). Hence $y' = -\frac{2}{3}$ at $(x, y) = (-1, 0)$. (1 pt for $y' = -2/3$.)

(b) We further differentiate the equation $4x + 6y + 6xy' + 10yy' = 0$ and obtain

$$4 + 6y' + 6y' + 6xy'' + 10(y')^2 + 10yy'' = 0. \qquad \text{(3 pts)}$$

At $(x, y) + (-1, 0)$, $y' = -2/3$, the above equation becomes $4 - 4 - 4 - 6y'' + \frac{40}{9} = 0$.
Hence $y'' = \frac{2}{27}$. (1 pt for $y'' = 2/27$.)

(c) The linearization of $f(x)$ at $x = -1$ is

$$L(x) = f(-1) + f'(-1)(x + 1) = -\frac{2}{3}(x + 1). \qquad \text{(1 pt)}$$

Hence $f(-1.03) \approx L(-1.03) = 0.02$. (1 pt for $f(-1.03) \approx -0.02$.)

(d) Since $f''(-1) = 2/27 > 0$ and $f''(x)$ is continuous neat $x = -1$, we know that $f''(x) > 0$ near $x = -1$. Hence, $y = f(x)$ is concave upward near $x = -1$ and the graph of $f(x)$ is above tangent lines. Therefore $f(-1.03) > L(-1.03)$.
(1 pt for $f''(x) > 0$ near $x = -1$.
1 pt for $f(-1.03) > L(-1.03)$.)

3. (a) (6 pts) Evaluate the limit

$$\lim_{x\to 0^+} \frac{x+\sqrt{x}}{2x+\sin x}.$$

(b) (2 pts) Consider the limit

$$\lim_{x\to \infty} \frac{x+\sqrt{x}}{2x+\sin x}.$$

Explain why this limit cannot be evaluated with l'Hospital's Rule.

(c) (6 pts) Evaluate the limit

$$\lim_{x\to \infty} \frac{x+\sqrt{x}}{2x+\sin x}.$$

Solution:

(a) Solution 1: Note that $\lim_{x\to 0} x+\sqrt{x} = 0$ and $\lim_{x\to 0} 2x+\sin x = 0$. Hence $\lim_{x\to 0} \frac{x+\sqrt{x}}{2x+\sin x}$ is a $\frac{0}{0}$ indeterminate form. By l'Hospital's Rule, we have

$$\lim_{x\to 0} \frac{x+\sqrt{x}}{2x+\sin x} \overset{\frac{0}{0} L'H}{=} \lim_{x\to 0} \frac{1+\frac{1}{2\sqrt{x}}}{2+\cos x}.$$

Observe that $\lim_{x\to 0} 1+\frac{1}{2\sqrt{x}} = \infty$ and $\lim_{x\to 0} 2+\cos x = 3$. Hence

$$\lim_{x\to 0} \frac{x+\sqrt{x}}{2x+\sin x} \overset{\frac{0}{0} L'H}{=} \lim_{x\to 0} \frac{1+\frac{1}{2\sqrt{x}}}{2+\cos x} = \infty.\text{不存在 } -1$$

(1 pt for applying l'Hospital's Rule.
1 pt for $\frac{d}{dx}(x+\sqrt{x})$. 1 pt for $\frac{d}{dx}(2x+\sin x)$. 1 pt for $(\frac{0}{0})$. 2 pts for the final answer.)

Solution 2:

$$\lim_{x\to 0} \frac{x+\sqrt{x}}{2x+\sin x} = \lim_{x\to 0} \frac{1+\frac{1}{\sqrt{x}}}{2+\frac{\sin x}{x}}$$

Observe that $\lim_{x\to 0} 1+\frac{1}{\sqrt{x}} = \infty$ and $\lim_{x\to 0} 2+\frac{\sin x}{x} = 3$. Hence

$$\lim_{x\to 0} \frac{x+\sqrt{x}}{2x+\sin x} = \lim_{x\to 0} \frac{1+\frac{1}{\sqrt{x}}}{2+\frac{\sin x}{x}} = \infty.$$

(2 pts for dividing the numerator and denominator by x,
2 pts for using the special limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$. 2 pts for the final answer.)

(b) $\lim_{x\to\infty} \frac{x+\sqrt{x}}{2x+\sin x}$ is a $\frac{\infty}{\infty}$ indeterminate form. However, after differentiating both the numerator and denominator, the limit $\lim_{x\to\infty} \frac{x+\frac{1}{2\sqrt{x}}}{2+\cos x}$ dose not exist because $\lim_{x\to\infty} 1+\frac{1}{2\sqrt{x}} = 0$ but the denominator $2+\cos x$ oscillates as $x\to\infty$. Hence we can not apply l'Hospital's Rule.

(2 pts for stating that $\lim_{x\to\infty} \frac{x+\frac{1}{2\sqrt{x}}}{2+\cos x}$ dose not exist.)

(c)

$$\lim_{x\to\infty} \frac{x+\sqrt{x}}{2x+\sin x} = \lim_{x\to\infty} \frac{1+\frac{1}{\sqrt{x}}}{2+\frac{\sin x}{x}}$$

Note that $-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$ for $x>0$ and $\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$. Hence by the squeeze theorem, $\lim_{x\to\infty} \frac{\sin x}{x} = 0$. Moreover, $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$. Hence, by limit laws,

$$\lim_{x\to\infty} \frac{x+\sqrt{x}}{2x+\sin x} = \lim_{x\to\infty} \frac{1+\frac{1}{\sqrt{x}}}{2+\frac{\sin x}{x}} = \frac{1+0}{2+0} = \frac{1}{2}.$$

(2 pts for dividing the numerator and denominator by x,
3 pts for deriving $\lim_{x\to\infty} \frac{\sin x}{x} = 0$. 1 pts for the final answer.)

4. Consider the function

$$f(x) = |x|(\cos x - 1).$$

(a) (4 pts) Find $f'(x)$ for $x \neq 0$. (Hint: separate $x > 0$ and $x < 0$ cases.)

(b) (4 pts) Find $f'(0)$ with the definition of derivatives.

(c) (4 pts) Does $f''(0)$ exist? If yes, find it.

Solution:

(a) For $x > 0$, $f(x) = x(\cos x - 1)$. Then we have

$$f'(x) = (\cos x - 1) + x(-\sin x) = -x\sin x + \cos x - 1. \ (2\%)$$

For $x < 0$, $f(x) = -x(\cos x - 1)$. Then we have

$$f'(x) = -(\cos x - 1) - x(-\sin x) = x\sin x - \cos x + 1. \ (2\%)$$

(b)

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|(\cos h - 1)}{h}. \ (2\%)$$

Since

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h(\cos h - 1)}{h} = \lim_{h\to 0^+} (\cos h - 1) = 0 \ (1\%)$$

and

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-h(\cos h - 1)}{h} = \lim_{h\to 0^-} -(\cos h - 1) = 0, \ (1\%)$$

we have $f'(0) = 0$.

(c) By definition,

$$f''(0) = \lim_{h\to 0} \frac{f'(h) - f'(0)}{h}. \ (1\%)$$

We compute that

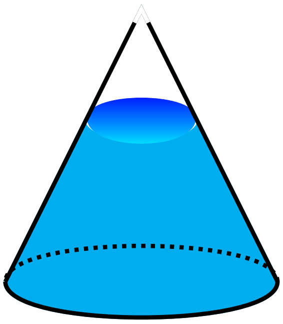
$$\lim_{h\to 0^+} \frac{f'(h) - f'(0)}{h} = \lim_{h\to 0^+} \frac{-h\sin h + \cos h - 1}{h} = \lim_{h\to 0^+} (-\sin h + \frac{\cos h - 1}{h}) = 0 \ (1\%)$$

and

$$\lim_{h\to 0^-} \frac{f'(h) - f'(0)}{h} = \lim_{h\to 0^-} \frac{h\sin h - \cos h + 1}{h} = \lim_{h\to 0^+} (\sin h - \frac{\cos h - 1}{h}) = 0 \ (1\%)$$

So we have $f''(0) = 0$. (1%)

5. (8 pts) A conical tank is being filled with water at a constant rate of 3 cubic feet per minute (see figure). The tank has height of 10 feet and a base radius of 5 feet. How fast is the water level rising when the water is 6 feet deep?



Solution:

Set h is the level of water, r is the radius of the top of the water and V is the volume of the water. (1%) Then we have

$$V = \frac{\pi}{3} \cdot 5^2 \cdot 10 - \frac{\pi}{3} \cdot r^2(10-h)$$

and $\frac{dV}{dt} = 3$. (2%) Moreover, r and h satisfy

$$\frac{r}{5} = \frac{10-h}{10} \Rightarrow r = \frac{10-h}{2} \text{ (1\%)}$$

This shows us the volume

$$V = V(h) = \frac{250\pi}{3} - \frac{\pi}{3} \cdot \left(\frac{10-h}{2}\right)^2 (10-h) = \frac{250\pi}{3} - \frac{\pi}{12}(10-h)^3. \text{ (1\%)}$$

So we have $\left.\frac{dV}{dh}\right|_{h=6} = \left.\frac{\pi}{4}(10-h)^2\right|_{h=6} = 4\pi$. (1%). Therefore, we obtain that

$$\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dh}} = \frac{3}{4\pi}. \text{ (2\%)}$$

6. Suppose that $f(x)$ is continuous on $[a,b]$, $f(a) = f(b) = 0$, and $f''(x) < 0$ for all $x \in (a,b)$.

   (a) (3 pts) Show that for any $x_1 < x_2$ in (a,b), $f'(x_1) > f'(x_2)$.

   (b) (3 pts) By Rolle's Theorem, there exists $c \in (a,b)$ with $f'(c) = 0$. Show that f has exactly one critical number.

   (c) (3 pts) Show that the absolute minimum value(global minimum value) of f over the interval $[a,b]$ is 0.

   (d) (3 pts) Show that $f(x) > 0$ for all $x \in (a,b)$.

Solution:

First we need to observe that f is differentiable over (a,b), $f'(x)$ exist for all $x \in (a,b)$, f' is continuous and differentiable over (a,b).

(a) Consider the interval $[x_1, x_2]$. f' is continuous over $[x_1, x_2]$ and differentiable over (x_1, x_2). Therefore by the Mean Value Theorem,

$$f'(x_2) - f'(x_1) = f''(x_0)(x_2 - x_1) \quad \text{for some} \quad x_0 \in (x_1, x_2).$$

Since $(x_2 - x_1) > 0$ and $f''(x_0) < 0$, we can conclude that $f'(x_1) > f'(x_2)$.

(b) Suppose that f has more than one critical number in (a,b). f is differentiable over (a,b) so all critical numbers satisfy $f'(x) = 0$. Then we can find $c_1 < c_2$ in (a,b) with $f'(c_1) = 0$ and $f'(c_2) = 0$. Either from (a) or by MVT over (c_1, c_2), we get a contradiction. Hence f has exactly one critical number.

(c) From the result in (b), the closed interval method states that the absolute extrema only occur at $x = a, b, c$, where c is the unique critical point from (b). Since $f''(c) < 0$ as given in the problem, the second derivative test states that the point $(c, f(c))$ is a local maximum point, which cannot be the absolute minimum point. Therefore the absolute minimum value is $f(a) = f(b) = 0$.

Or use (a) and (b) to say the function is increasing over (a,c) and decreasing over (c,b). Or prove by the method of contradiction and use the mean value theorem.

(d) Suppose that $f(d) \leq 0$ for some $d \in (a,b)$. Use the mean value theorem over (a,d) and (d,b) to conclude that there are $c_1 \in (a,d)$ with $f'(c_1) \leq 0$ and $c_2 \in (d,b)$ with $f'(c_2) \geq 0$. This contradicts with the result in (a).

Or use a similar argument in (c) (just the result of (c) is not enough). Or discuss strictly increasing and strictly decreasing. ☐

Grading:

Students can prove the 4 parts in any order, so they can use results they proved (or thought they proved). We would assume the order (a) (b) (c) (d) unless specified.

For each part, (1 pt) for proving 1 fact toward the result. (2 pts) for completing the proof, and (-1 pt) here for each skipped step or logical error. (yes, students can get 2 pts even if they show a very clear logical error)

The first point can be earned as long as the student says something useful, such as f' is continuous, MVT, EVT, closed interval method, setting up a proof by contradiction, make observations, or quoting other parts of this problem to be used.

No points for copying the problem down or just drawing a picture with no explanation. Grader can decide if any mistake is very minor and just needs (-0.5 pts), or if any mistake is critical and should get (-2 pts). Do not take points off just because the student did not write down the same steps as the solution. We understand that students make lots of logical mistakes, so the grader must be careful, read their whole work, and point out mistakes to deduct points carefully.

7. Let $f(x) = (x-6)e^{-\frac{1}{x}}$.

(a) (6 pts) Compute $f'(x)$. Find the intervals of increase or decrease.

(b) (6 pts) Compute $f''(x)$. Find the intervals of concavity.

(c) (2 pts) Compute $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

(d) (4 pts) Let $m = \lim_{x\to\infty} \frac{f(x)}{x}$. Compute $\lim_{x\to\infty} [f(x) - mx]$. Also evaluate limits to $-\infty$ to find the two slant asymptotes.

(e) (2 pts) Sketch the graph $y = f(x)$ and its asymptotes. Label your sketch with answers from (a)-(d).

Solution:

(a) (Step 1) (1%) $f'(x) = e^{\frac{-1}{x}} + (x-6)e^{\frac{-1}{x}} \cdot x^{-2} = e^{\frac{-1}{x}}(\frac{x^2+x-6}{x^2})$.

(Step 2) (2%) Solve for critical points of f':

$$-3, 0(1 \text{ pt}), \text{and } 2$$

(Step 3) (2%) Determine signs of each interval:

-	-3	-	0	-	2	-
+	0	−	DNE	−	0	+

(Step 4) (1%) Intervals of increase: $(-\infty, -3) \cup (2, \infty)$.
Intervals of decrease: $(-3, 0) \cup (0, 2)$

(b) (1%) $f''(x) = e^{\frac{-1}{x}} \cdot \frac{1}{x^2} \cdot \left(\frac{x^2+x-6}{x^2}\right) + e^{\frac{-1}{x}}\left(\frac{(2x+1)x^2 - (x^2+x-6)\cdot 2x}{x^4}\right) = e^{\frac{-1}{x}} \frac{13x-6}{x^4}$.

(2%) Set $f''(x) = 0$ (1 pt) and correctly solve zeros of f'' (1 pt): $x = \frac{6}{13}$.

(2%) Determine sign of each interval:

-	0	-	$\frac{6}{13}$	-
−	−	−	−	+

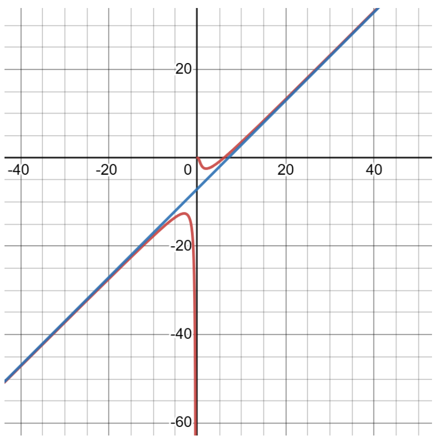
(1 %) Interval of concave: $(-\infty, 0) \cup (0, \frac{6}{13})$.

Interval of convex: $(\frac{6}{13}, \infty)$.

(c) (Any sign of applying L'Hopital gives 1 pt) $\lim_{x\to 0-} f(x) = -\infty$ (0.5 pt) and $\lim_{x\to 0^+} f(x) = 0$ (0.5 pt).

(d) $L = \lim_{x\to\infty} \frac{f(x)}{x} = 1$(1 pt), and $\lim_{x\to\infty} f(x) - x = -7$.(1 pt). The answers and points are the same for $x \to -\infty$.

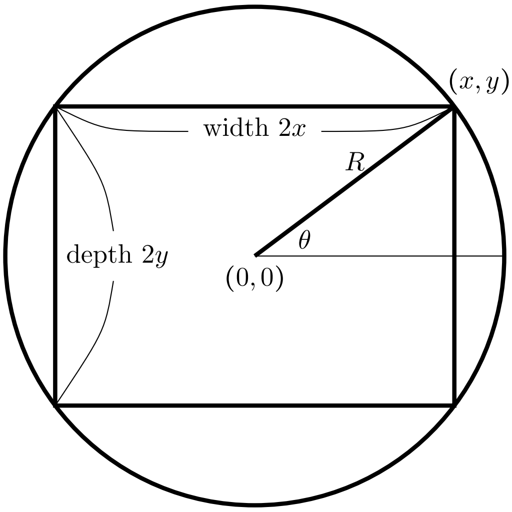
(e) Please create a graph based on the information provided in the previous answers. A 0.5-point deduction will be applied to any part of the graph that does not align with the explanations given above



8. A wooden beam of rectangular cross section is to be cut out of a circular log of radius R (see figure). The dimensions of the beam is determined by θ.

(a) (4 pts) Find the θ value that will maximize the area of the cross section.

(b) (8 pts) If the *stiffness* of the beam is proportional to the width and the cube of the depth, given by $K(2x)(2y)^3$, $K > 0$ is a constant, find the θ value that will result in the stiffest beam (maximize stiffness).



Solution:

(a) Let $f(\theta)$ be the cross sectional area function. We can see from the figure that $x = R\cos\theta$ and $y = R\sin\theta$.

$$f(\theta) = 4xy = 4R^2\cos\theta\sin\theta = 2R^2\sin(2\theta)$$

From the figure we can see that the domain for f is $[0, \pi/2]$. We can find f' and then find the critical numbers but it is also clear that $\sin(2\theta)$ has a maximum value at $\theta = \pi/4$. The θ value that maximizes the area of the cross section is $\pi/4$ and the maximum cross sectional area is $2R^2$.

(b) Let $g(\theta)$ be the stiffness of the beam.

$$g(\theta) = 16KR^4\cos\theta\sin^3\theta\ ,\quad \theta \in [0, \pi/2].$$

$$g'(\theta) = 16KR^4\left(-\sin^4\theta + 3\sin^2\theta\cos^2\theta\right) = 16KR^4\sin^2\theta\cos^2\theta\left(3 - \tan^2\theta\right)$$

Since g is equal to zero at $\theta = 0$ and $\theta = \pi/2$, the maximum must occur in $(0, \pi/2)$. The derivative is equal to zero when $\tan\theta = \sqrt{3}$, $\theta = \pi/3$.

Over $(0, \pi/3)$, $\tan^2\theta < 3$, so $g'(\theta) > 0$, increasing.

Over $(\pi/3, \pi/2)$, $\tan^2\theta > 3$, so $g'(\theta) < 0$, decreasing.

Or use the closed interval method. $g(\pi/3) = 3\sqrt{3}KR^4 > 0$.

The θ value that will result in the stiffest beam is $\pi/3$. □

Grading:

(a) (2 pts) for finding the area function. (2 pts) for the answer and explaining why it is the maximum.

(b) (2 pts) for the stiffness function. (1 pt) for derivative. (3 pts) for solving for critical numbers. (2 pts) for the answer and explaining why it is the maximum.

Domain is part of the explanation. If a mistake is repeated in (a) and (b), only take points off once. Minor mistakes get (-1 pts) and major mistakes get (-2 pts).