

1. Evaluate the following limits

(a) (5 pts) $\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{\sqrt{3x^2 + 4} - \sqrt{x^2 + 4}}$

(b) (5 pts) $\lim_{x \rightarrow \infty} \left(\frac{x+a}{x-a} \right)^x$ (Express your answer in terms of a .)

(c) (5 pts) $\lim_{x \rightarrow -\infty} \frac{\lceil 2x + \sqrt{|x|} \rceil}{\sqrt{x^2 + 1}}$ where $\lceil \cdot \rceil$ denotes the greatest integer function.
(Hint. For any $y \in \mathbb{R}$, we have $y - 1 < \lceil y \rceil \leq y$.)

Solution:

(a)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{\sqrt{3x^2 + 4} - \sqrt{x^2 + 4}} &= \lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{2x^2} \cdot (\sqrt{3x^2 + 4} + \sqrt{x^2 + 4}) \\ &= \lim_{x \rightarrow 0} \frac{2\sin^2(x)}{2x^2} \cdot (\sqrt{3x^2 + 4} + \sqrt{x^2 + 4}) \\ &= 1 \cdot (2 + 2) = 4 \end{aligned}$$

Marking Scheme for Q1(a).

1M for attempting to rationalize

1M for correct rationalization

2M for computing the limit $\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{x^2}$

1M for correct answer

Remarks.

If a student attempts to compute this limit by L'Hospital's rule directly, he/she can earn at most 2M for any incomplete/incorrect computations ; and full marks only if the answer is correct.

(b) **(Method 1 - use L'Hospital's rule)**

Let $y = \left(\frac{x+a}{x-a} \right)^x$. Then

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} \frac{\ln(x+a) - \ln(x-a)}{\frac{1}{x}} \\ &\stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x+a} - \frac{1}{x-a}}{-\frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{2ax^2}{x^2 - a^2} \\ &= 2a \end{aligned}$$

Therefore, we conclude that $\lim_{x \rightarrow \infty} y = e^{2a}$.

(Method 2 - express 'e' as a limit)

If $a = 0$, then the limit equals 1 obviously. Suppose $a \neq 0$. Then we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{x+a}{x-a} \right)^x &= \lim_{x \rightarrow \infty} \left(1 + \frac{2a}{x-a} \right)^x \\ &= \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\left(\frac{x-a}{2a} \right)} \right)^x \\ &= \lim_{x \rightarrow \infty} \left(\left(1 + \frac{1}{\left(\frac{x-a}{2a} \right)} \right)^{\frac{x-a}{2a}} \right)^{2a} \cdot \left(1 + \frac{1}{\left(\frac{x-a}{2a} \right)} \right)^a \\ &= e^{2a} \cdot (1+0)^a = e^{2a}. \end{aligned}$$

Marking Scheme for Q1(b).

2M for either attempts (either rearrange into $\frac{0}{0}$ form or rearrange into $\lim_{z \rightarrow \pm\infty} \left(1 + \frac{1}{z}\right)^z$.

2M for either applying (correctly) L'Hospital's rule or the limit 'e'.

1M for correct answer

(c) Since

$$2x + \sqrt{|x|} - 1 \leq \llbracket 2x + \sqrt{|x|} \rrbracket \leq 2x + \sqrt{|x|},$$

we then have

$$\frac{2x + \sqrt{|x|} - 1}{\sqrt{x^2 + 1}} \leq \frac{\llbracket 2x + \sqrt{|x|} \rrbracket}{\sqrt{x^2 + 1}} \leq \frac{2x + \sqrt{|x|}}{\sqrt{x^2 + 1}}.$$

As $\lim_{x \rightarrow -\infty} \frac{2x + \sqrt{|x|}}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow -\infty} \frac{2 - \sqrt{-1/x}}{-\sqrt{1 + 1/x^2}} = -2$ and $\lim_{x \rightarrow -\infty} \frac{2x + \sqrt{|x|} - 1}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow -\infty} \frac{2 - \sqrt{-1/x} - 1/x}{-\sqrt{1 + 1/x^2}} = -2$, the

Squeeze Theorem implies that $\lim_{x \rightarrow -\infty} \frac{\llbracket 2x + \sqrt{|x|} \rrbracket}{\sqrt{x^2 + 1}} = -2$.

Marking Scheme for Q1(c).

1M for writing down the correct bounds for $\llbracket 2x + \sqrt{|x|} \rrbracket$.

3M for computing one of the limits from lower/upper bounds correctly (with a valid argument) (at most 2M for making a sign error on the answer)

1M for applying Squeeze Theorem (correctly)

Remark. To graders, please check :

- -0.5M for anybody writing $f(x) < g(x)$ implies $\lim f(x) < \lim g(x)$.
- -0.5M for making any sign errors (with correct answer though)

2. Compute the following derivatives.

(a) (6 pts) Suppose that $y^x + x \cos(y^2) + y = 2$. Find $\left. \frac{dy}{dx} \right|_{(0,1)}$.

(b) (6 pts) Let $f(x) = \frac{x^{x^2}(x + \ln x)}{\sqrt[3]{x + \sqrt{x}}}$. Find $f'(1)$.

Solution:

(a)

$$y^x + x \cos(y^2) + y = 2$$

$$e^{x \ln y} + x \cos(y^2) + y = 2$$

$$e^{x \ln y} \left(\ln y + \frac{x}{y} y' \right) + \cos(y^2) - 4xy \cos(y^2) \sin(y^2) y' + y' = 0$$

$$e^0 (\ln 1 + 0 \cdot y') + \cos(1) - 0 \cdot y' + y' = 0$$

$$y' = -\cos 1$$

(b)

$$f(x) = \frac{x^{x^2}(x + \ln x)}{\sqrt[3]{x + \sqrt{x}}}$$

$$\ln f(x) = \ln \left(\frac{x^{x^2}(x + \ln x)}{\sqrt[3]{x + \sqrt{x}}} \right)$$

$$\ln f(x) = x^2 \ln x + \ln(x + \ln x) - \frac{1}{3} \ln(x + \sqrt{x})$$

$$\frac{f'(x)}{f(x)} = 2x \ln x + x + \frac{1 + 1/x}{x + \ln x} - \frac{1 + \frac{1}{2\sqrt{x}}}{3(x + \sqrt{x})}$$

$$f(1) = \frac{1}{\sqrt[3]{2}}$$

$$f'(1) \cdot \sqrt[3]{2} = 0 + 1 + \frac{2}{1} - \frac{1 + \frac{1}{2}}{3(1 + 1)}$$

$$f'(1) = \frac{11}{4\sqrt[3]{2}}$$

□

Grading:

Each part is worth (6 pts) and students get (-2 pts) for each mistake. Grader can decide if a mistake is very minor and do (-1 pt) instead.

3. Ray gets stuck at his homework on calculus so he consults the ChatGPT, an AI-powered language model, for help.

(a) The following is extracted from their conversations.



Is $f(x) = \cos(x^{2/3})$ differentiable at $x=0$?



To determine whether the function $f(x) = \cos(x^{2/3})$ is differentiable at $x = 0$, we need to examine its derivative at that point.

First, let's find the derivative of $f(x) = \cos(x^{2/3})$ with respect to x :

Using the chain rule, we can find the derivative as follows:

$$f'(x) = -\sin(x^{2/3}) \cdot \frac{d}{dx}(x^{2/3})$$

Now, let's find the derivative of $x^{2/3}$:

$$\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{-1/3}$$

So, we have:

$$f'(x) = -\sin(x^{2/3}) \cdot \frac{2}{3}x^{-1/3}$$

Now, we can evaluate $f'(x)$ at $x = 0$:

$$f'(0) = -\sin(0^{2/3}) \cdot \frac{2}{3} \cdot 0^{-1/3}$$

Since $\sin(0) = 0$ and $0^{-1/3}$ is undefined, $f(x) = \cos(x^{2/3})$ is not differentiable at $x = 0$.

(i) (2 pts) By referring to the boxed part, point out precisely where the argument of ChatGPT is incorrect.

(ii) (4 pts) Determine, with a correct argument, whether the function $f(x) = \cos(x^{2/3})$ is differentiable at $x = 0$.

(iii) (4 pts) If $f(x) = \cos(x^{2/3})$, is $f'(x)$ continuous at $x = 0$? Justify your answer.

(b) (2 pts) Ray further asks ChatGPT the following question :



Is it true that if $f(x)$ is strictly increasing, then $f'(x) > 0$ for all x .



Yes, it is true that if a function $f(x)$ is strictly increasing, then its derivative $f'(x)$ is greater than 0 for all x in the interval where $f(x)$ is defined.

By offering an explicit counter-example, explain why ChatGPT's claim is false.

Solution:

(a) (i) We accept the following answers.

- The definition of f at $x = 0$ is solely determined by the existence of the limit $\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$ and is not related to either the domain of $f'(x)$ for $x \neq 0$, nor the limit $\lim_{x \rightarrow 0} f'(x)$.
- 'The domain of $f'(x)$ for $x \neq 0$ does not extend to $x = 0$ ' does not imply f itself is not differentiable at $x = 0$.
- 'The properties of $f'(x)$ for $x \neq 0$ ' has no relations with the differentiability of f at $x = 0$ because differentiability itself is a 'local property' (i.e. defined point-wise).

Marking Scheme .

2M for perfect answers;

1M for those who managed to address the issue, but not entirely accurate;

0M for any misunderstanding in differentiability

(ii) To determine the differentiability of $f(x)$ at $x = 0$, consider the limit

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\cos(h^{2/3}) - 1}{h} = \lim_{h \rightarrow 0} \frac{\cos(h^{2/3}) - 1}{h^{4/3}} \cdot h^{1/3} = -\frac{1}{2} \cdot 0 = 0.$$

In particular the limit exists so $f(x)$ is differentiable at $x = 0$ with $f'(0) = 0$.

Marking Scheme .

2M for the correct definition of differentiability;

1M for the correct evaluation of the limit

1M for the correct conclusion/overall coherence of argument.

- (iii) To determine the continuity of $f'(x)$ at $x = 0$, consider the limit

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} -\sin(x^{2/3}) \cdot \frac{2}{3} x^{-1/3} = \lim_{x \rightarrow 0} -\frac{2 \sin(x^{2/3})}{3 x^{2/3}} \cdot x^{1/3} = -\frac{2}{3} \cdot 1 \cdot 0 = 0 = f'(0).$$

Therefore, $f'(x)$ is continuous at $x = 0$.

Marking Scheme .

1M for the correct definition of continuity of $f'(x)$ at $x = 0$;

2M for the correct evaluation of the limit $\lim_{x \rightarrow 0} f'(x)$

1M for correct conclusion/overall coherence of argument.

- (b) There are many counter-examples, such as $f(x) = x^3$: this is a strictly increasing function but it is untrue that $f'(x) > 0$ for all real number x .

Marking Scheme .

All or nothing.

4. Consider the function $f(x) = e^{2x} + \tan^{-1}(x)$ for $x \in \mathbb{R}$.

- (a) (4 pts) Find $\lim_{x \rightarrow -\infty} f(x)$ and $\lim_{x \rightarrow \infty} f(x)$. Find the range of $f(x)$.
- (b) (2 pts) Show that $f(x)$ is one-to-one.
- (c) By (b), $f(x)$ has an inverse function. Let $g(x) = f^{-1}(x)$ be the inverse function of $f(x)$. You may use without proof that g is smooth everywhere (i.e. g is infinitely differentiable).
- (i) (6 pts) Write down the linearization of $g(x)$ at $x = 1$. Use this to estimate the value of $g(0.97)$.
- (ii) (4 pts) Find $g''(1)$. Is the estimation in part (c) greater than or less than the actual value of $g(0.97)$?

Solution:

- (a) $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} e^{2x} + \lim_{x \rightarrow -\infty} \tan^{-1} x = 0 + \left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}$ (2 pts for the final answer).
 $\lim_{x \rightarrow \infty} f(x) = \infty$ (1 pt for the final answer). (-0.5 for only writing DNE.)

Because $f(x)$ is continuous and by the intermediate value theorem, $f(x)$ can obtain any value in $(-\frac{\pi}{2}, \infty)$.

On the other hand, $e^{2x} > 0$ and $\tan^{-1} x > -\frac{\pi}{2}$ for all $x \in \mathbb{R}$. Thus $f(x) > -\frac{\pi}{2}$ for all $x \in \mathbb{R}$.

Hence the range of $f(x)$ is $(-\frac{\pi}{2}, \infty)$. (1 pt for the final answer: the range is $(-\frac{\pi}{2}, \infty)$) (-0.5 pt for wrong notation of intervals.)

- (b) Because $f'(x) = 2e^{2x} + \frac{1}{1+x^2} > 0$, $f(x)$ is strictly increasing by the Increasing/Decreasing Test (or the Mean Value Theorem). Hence $f(x)$ is one-to-one.
 (1 pt for computing $f'(x)$. 1 pt for deriving that $f(x)$ is one-to-one. Students can prove this by Rolle's Theorem.)

- (c) (i) Because $f(0) = 1$, we know that $g(1) = 0$. (1 pt for $g(1)$)

Differentiating $f(g(x)) = x$, we obtain $f'(g(x))g'(x) = 1$ which implies that $g'(x) = \frac{1}{f'(g(x))}$.

(1 pt)

$$\text{Thus } g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)}. \quad (1 \text{ pt})$$

Since $f'(0) = 2e^0 + \frac{1}{1+0} = 3$, we have $g'(1) = \frac{1}{3}$. (1 pt)

The linearization of $g(x)$ at $x = 1$ is

$$L(x) = g(1) + g'(1)(x - 1) = 0 + \frac{1}{3}(x - 1) = \frac{1}{3}(x - 1). \quad (1 \text{ pt for the definition})$$

Hence $g(0.97) \approx L(0.97) = -0.01$. (1 pt)

- (ii) By differentiating $f'(g(x))g'(x) = 1$ we obtain $f''(g(x))(g'(x))^2 + f'(g(x))g''(x) = 0$. (1 pt)

$$\text{At } x = 1, f''(0) \cdot \frac{1}{9} + f'(0) \cdot g''(1) = 0.$$

Since $f''(x) = 4e^{2x} - \frac{2x}{(1+x^2)^2}$, we have $f''(0) = 4$ (1 pt)

$$\text{Thus } g''(1) = -\frac{4}{27}. \quad (1 \text{ pt})$$

Because $g''(x) < 0$ near $x = 1$, the tangent line at $x = 1$ lies above the graph of $g(x)$. Hence the linear approximation $L(0.97)$ is greater than $g(0.97)$. (1 pt)

5. (a) (5 pts) Suppose that $f(x)$ and $g(x)$ are continuous on $[a, b]$ and differentiable on (a, b) , and $g'(x) \neq 0$ for all $x \in (a, b)$. Then there is some $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Hint. Consider the function $h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot (g(x) - g(a))$.

Remark. In literature, this is called the Cauchy's Mean Value Theorem.

- (b) (6 pts) Suppose f is twice differentiable on an interval containing a . Let

- $L(x)$ be the linearization of $f(x)$ at $x = a$;
- $P(x) = \frac{f(x) - L(x)}{(x - a)^2}$.

By applying the result of (a) to $P(x)$, prove that there is some c strictly between a and x such that

$$f(x) - L(x) = \frac{f''(c)}{2}(x - a)^2.$$

- (c) (3 pts) Let $f(x) = \tan x$. Let $L(x)$ be the linearization of $f(x)$ at $x = \frac{\pi}{4}$. Prove that

$$\frac{4}{3\sqrt{3}} \left(x - \frac{\pi}{4}\right)^2 < f(x) - L(x) < 4\sqrt{3} \left(x - \frac{\pi}{4}\right)^2 \quad \text{for } \frac{\pi}{6} \leq x \leq \frac{\pi}{3}.$$

Solution:

- (a) $h(x)$ is continuous on $[a, b]$ and differentiable on (a, b) with $h'(x) = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x)$.

Moreover, $h(a) = h(b) = 0$. Hence by Rolle's Theorem, there is some $c \in (a, b)$ such that

$$h'(c) = 0 = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c)$$

which means that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(1 pt for stating that $h(x)$ is continuous on $[a, b]$ and differentiable on (a, b) .)

1 pt for checking that $h(a) = h(b) = 0$.

1 pt for $h'(x)$.

2 pts for applying Rolle's Theorem and obtain the result.)

- (b) The linearization of $f(x)$ at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a). \quad (1 \text{ pt for } L(x))$$

Hence $f(x) - L(x) = f(x) - f(a) - f'(a)(x - a)$ and $f'(x) - L'(x) = f'(x) - f'(a)$.

(1 pt for computing $f'(x) - L'(x)$)

Note that $f(x) - L(x)$ and $(x - a)^2$ are differentiable on an interval containing a and the derivative of $(x - a)^2$ is nonzero for $x \neq a$. Hence for any x near a , we can apply Cauchy's Mean Value Theorem on $P(x) = \frac{f(x) - L(x)}{(x - a)^2}$ on the interval between a and x . Therefore there is some d between a and x such that

$$\frac{f(x) - L(x)}{(x - a)^2} = \frac{f'(d) - L'(d)}{2(d - a)} = \frac{f'(d) - f'(a)}{2(d - a)}.$$

(2 pts for applying Cauchy's Mean Value Theorem.)

Then we further apply the Mean Value Theorem on the rightest term and conclude that there is some c between d and a such that

$$\frac{f(x) - L(x)}{(x - a)^2} = \frac{f'(d) - f'(a)}{2(d - a)} = \frac{f''(c)}{2}.$$

This means that there is some c between a and x such that $f(x) - L(x) = \frac{f''(c)}{2}(x - a)^2$.
(2 pts for applying the Mean Value Theorem.)

(c) For $x \in [\frac{\pi}{6}, \frac{\pi}{3}]$, there is some c between x and $\frac{\pi}{4}$ such that $f(x) - L(x) = \frac{f''(c)}{2}$.

Since $x \in [\frac{\pi}{6}, \frac{\pi}{3}]$, we have $c \in (\frac{\pi}{6}, \frac{\pi}{3})$. Furthermore, $f''(x) = 2 \sec^2 x \tan x$ is increasing on $[\frac{\pi}{6}, \frac{\pi}{3}]$.

Hence we conclude that $f''(\frac{\pi}{6}) = \frac{8}{3\sqrt{3}} < f''(c) < f''(\frac{\pi}{3}) = 8\sqrt{3}$ and obtain

$$\frac{4}{3\sqrt{3}}(x - \frac{\pi}{4})^2 < f(x) - L(x) = \frac{f''(c)}{2} < 4\sqrt{3}(x - \frac{\pi}{4})^2.$$

(1 pt for $f''(x) = 2 \sec^2 x \tan x$.

2 pts for the maximum and minimum values of $f''(x)$ on $[\frac{\pi}{6}, \frac{\pi}{3}]$.)

6. A rectangular beam will be cut from a cylindrical log of radius 10 inches (see Figure 1).

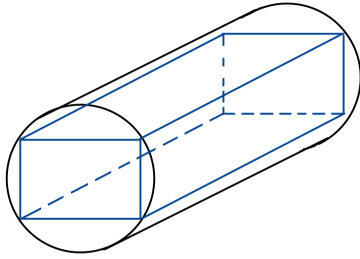


Figure 1

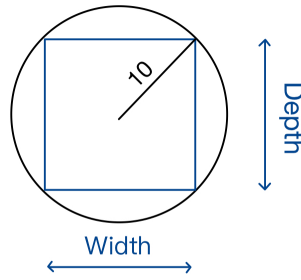


Figure 2

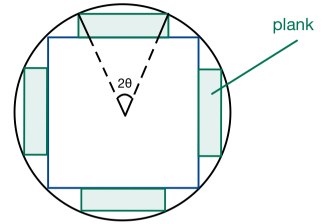


Figure 3

- (a) (i) (4 pts) Figure 2 shows the cross-section of the log. Prove that the beam of maximal cross-sectional area is a square.
- (ii) (6 pts) After cutting the square beam, four additional rectangular planks will be cut from the four sections of the remaining log (See Figure 3). Let 2θ be angle between the center of the log and the two outer vertices of a plank. Determine the value of θ that maximizes the cross-sectional area of the planks.
- (b) (5 pts) Suppose that the strength of a rectangular beam is proportional to the product of its width and the *square* of its depth (See Figure 2). Find the width and depth of the strongest beam that can be cut from the cylindrical log.

Solution:

- (a) (i) **Method I.** Suppose the angle at the center and between the top two vertices is 2α with $0 < \alpha < \pi/2$. Then the depth and the width of the rectangle are $20 \sin \alpha$ and $20 \cos \alpha$, and the area A is $400 \sin \alpha \cos \alpha$. We have the derivative

$$A' = 400(\cos^2 \alpha - \sin^2 \alpha) = 400(\cos \alpha + \sin \alpha)(\cos \alpha - \sin \alpha).$$

Thus the only critical point of A occurs at $\alpha = \pi/4$, in which case A is a square. Notice that $A' > 0$ for $0 < \alpha < \pi/4$ and $A' < 0$ for $\pi/4 < \alpha < \pi/2$. Therefore at $\alpha = \pi/4$, A achieves its absolute maximum by the first derivative test.

[Setup the function A : (+2); compute A' : (+1); solve critical point: (+1).]

Method II. Let the width and the depth of the rectangle be $2x$ and $2y$ with $0 < x < 10$. Then $x^2 + y^2 = 10^2$ and one has $y = \sqrt{100 - x^2}$. The area A is equal to $4xy = 4x\sqrt{100 - x^2}$ and one has the derivative

$$\begin{aligned} A' &= 4\sqrt{100 - x^2} + 4x \frac{-x}{\sqrt{100 - x^2}} \\ &= \frac{8}{\sqrt{100 - x^2}}(50 - x^2). \end{aligned}$$

Thus the only critical point of A occurs at $x = \sqrt{50}$, in which case A is a square. Notice that $A' > 0$ for $0 < x < \sqrt{50}$ and $A' < 0$ for $\sqrt{50} < x < 10$. Therefore at $x = \sqrt{50}$, A achieves its absolute maximum by the first derivative test.

[Setup the function A : (+2); compute A' : (+1); solve critical point: (+1).]

- (ii) Let the angle θ be given as in the figure ($0 < \theta < \pi/4$). Then the top rectangle has a width $20 \sin \theta$ and its depth equals $(10 \cos \theta - \sqrt{50})$. It has an area B equal to $200 \sin \theta (\cos \theta - 1/\sqrt{2})$. We have the derivative

$$\begin{aligned} \frac{B'}{200} &= \cos \theta \left(\cos \theta - \frac{1}{\sqrt{2}} \right) - \sin^2 \theta \\ &= 2 \cos^2 \theta - \frac{1}{\sqrt{2}} \cos \theta - 1 \\ &= 2 \left(\cos \theta - \frac{1 + \sqrt{17}}{4\sqrt{2}} \right) \left(\cos \theta - \frac{1 - \sqrt{17}}{4\sqrt{2}} \right), \end{aligned}$$

which equals zero when

$$\cos \theta = \frac{1 + \sqrt{17}}{4\sqrt{2}}, \quad \theta = \arccos \frac{1 + \sqrt{17}}{4\sqrt{2}}$$

($\theta = 0.43\dots$). More precisely, notice that in the range $0 < \theta < \pi/4$, $\cos \theta$ decreases from 1 to $1/\sqrt{2}$. One has $B' > 0$ for $(1 + \sqrt{17})/4\sqrt{2} < \cos \theta < 1$ and $B' < 0$ for $1/\sqrt{2} < \cos \theta < (1 + \sqrt{17})/4\sqrt{2}$. Thus B achieves its absolute maximal at the critical point by the first derivative test.

[Find the width and depth: (+1) and (+2); compute B' : (+1); solve critical point: (+2).]

(b) **Method I.** Use the setting as above. Here we want to maximize the function $C = \sin^2 \alpha \cos \alpha$ with

$$C' = 2 \sin \alpha \cos^2 \alpha - \sin^3 \alpha = \sin \alpha (2 \cos^2 \alpha - \sin^2 \alpha).$$

The critical point occurs when $2 \cos^2 \alpha = \sin^2 \alpha$, that is, $\cos \alpha = 1/\sqrt{3}$ and $\sin \alpha = \sqrt{2}/\sqrt{3}$. In this case, the width is $20/\sqrt{3}$ and the depth is $20\sqrt{2}/\sqrt{3}$.

[Setup the function C : (+2); compute C' : (+1); solve critical point: (+1); find width and depth: (+1).]

Method II. Use the setting as above. Here we want to maximize the function $C = xy^2 = x(100 - x^2)$ with

$$C' = 100 - 3x^2.$$

The critical point occurs when $x = 10/\sqrt{3}$. In this case, the width is $20/\sqrt{3}$ and the depth is $20\sqrt{2}/\sqrt{3}$.

[Setup the function C : (+2); compute C' : (+1); solve critical point: (+1); find width and depth: (+1).]

7. Consider the function $f(x) = \ln(e^{2x} - 5e^x + 6)$.
- (2 pts) Find the vertical asymptote(s) of $y = f(x)$.
 - (4 pts) Find the slant/horizontal asymptote(s) of $y = f(x)$.
 - (4 pts) Find $f'(x)$. Write down the interval(s) of increase and interval(s) of decrease of $y = f(x)$.
 - (3 pts) Find $f''(x)$. Determine the concavity of $y = f(x)$ and find (if any) point(s) of inflection.
 - (3 pts) Sketch the graph of $y = f(x)$. Indicate on your sketch (if any) the local extrema, inflection point(s) and asymptote(s).

Solution:

(a)

Vertical asymptotes only occur at discontinuities and the boundary of the domain.

Domain of f :

$$\begin{aligned} e^{2x} - 5e^x + 6 &> 0 \\ (e^x - 2)(e^x - 3) &> 0 \\ e^x &> 3 \text{ or } e^x < 2 \\ (-\infty, \ln 2) \cup (\ln 3, \infty) \\ \lim_{x \rightarrow (\ln 2)^-} f(x) &= -\infty \\ \lim_{x \rightarrow (\ln 3)^+} f(x) &= -\infty \end{aligned}$$

The vertical asymptotes of $y = f(x)$ are $x = \ln 2$ and $x = \ln 3$.

(b)

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{t \rightarrow \infty} \ln t = \infty, \quad \text{where } t = e^{2x} - 5e^x + 6 = (e^x - 2)(e^x - 3) \\ \lim_{x \rightarrow -\infty} f(x) &= \ln 6 \end{aligned}$$

There is a horizontal asymptote in the $-\infty$ direction: $y = \ln 6$.

There might be a slant asymptote in the ∞ direction.

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{\ln(e^{2x} - 5e^x + 6)}{x}$$

Use ∞/∞ l'Hospital's Rule

$$= \lim_{x \rightarrow \infty} \frac{2e^{2x} - 5e^x}{e^{2x} - 5e^x + 6} = \lim_{x \rightarrow \infty} \frac{2 - 5e^{-x}}{1 - 5e^{-x} + 6e^{-2x}} = 2$$

Now consider the limit of $f(x) - 2x$.

$$\lim_{x \rightarrow \infty} [f(x) - 2x] = \lim_{x \rightarrow \infty} \ln(1 - 5e^{-x} + 6e^{-2x}) = 0$$

There is a slant asymptote $y = 2x$ in the ∞ direction.

(c)

$$f'(x) = \frac{2e^{2x} - 5e^x}{e^{2x} - 5e^x + 6} = \frac{e^x(2e^x - 5)}{(e^x - 2)(e^x - 3)}$$

When $2e^x - 5 = 0$, $x = \ln(2.5)$ is not in the domain.

Interval of increase: $(\ln 3, \infty)$.

Interval of decrease: $(-\infty, \ln 2)$

(d)

$$\begin{aligned} f''(x) &= \frac{(4e^{2x} - 5e^x)(e^{2x} - 5e^x + 6) - (2e^{2x} - 5e^x)^2}{(e^{2x} - 5e^x + 6)^2} \\ &= \frac{4e^{4x} - 20e^{3x} + 24e^{2x} - 5e^{3x} + 25e^{2x} - 30e^x - 4e^{4x} + 20e^{3x} - 25e^{2x}}{(e^{2x} - 5e^x + 6)^2} \\ &= \frac{-e^x(5e^{2x} - 24e^x + 30)}{(e^{2x} - 5e^x + 6)^2} \end{aligned}$$

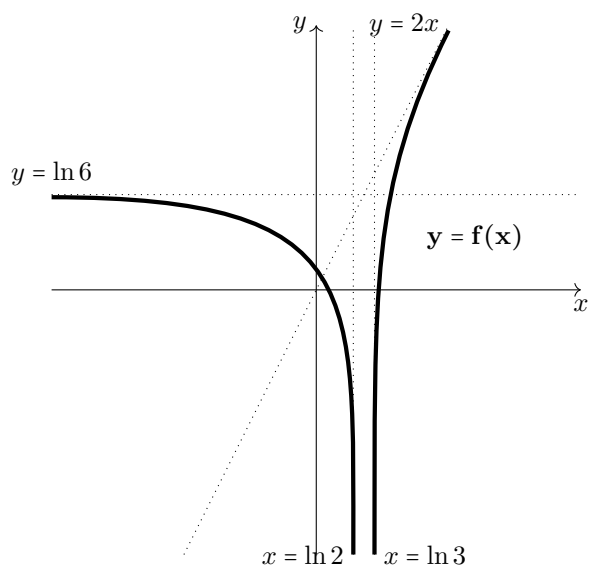
Check $(24)^2 - 4 \cdot 5 \cdot 30$ to find that $5e^{2x} - 24e^x + 30 > 0$ for all x .

Concave up interval: none.

Concave down intervals: $(-\infty, \ln 2) \cup (\ln 3, \infty)$

Therefore there are no inflection points.

(e)



□

Grading:

If students make any early mistakes, grader needs to use their answer for later parts and check consistency.

- (a) (1 pt) for domain and (1 pt) for limits/answer.
- (b) (2 pts) for each asymptote. (-1 pt) for any minor mistake.
- (c) (1 pt) for derivative. (2 pts) for sign chart. (1 pt) for interpreting the sign to inc/dec.
- (d) (1 pt) for derivative. (2 pts) for sign chart and answer.
- (e) Sketch must match all answers above. (-1 pt) for each mismatch. (-1 pt) for each unlabeled item.