112 微積分4-在經濟商管的應用 期考解答和評分標準

1. Let $A = \begin{pmatrix} 1 & -1 & 3 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 3 & -7 & 2 \end{pmatrix}$.

   (a) (4%) Find a row echelon form (REF) of A.

   (b) (2%) Find a basis of the row space of A and determine the rank of A.

   (c) (4%) Find values of a, b such that $\mathbf{v} = (a,\ 2,\ b,\ -1)$ belongs to the row space of A.

Solution:

(a) $A = \begin{pmatrix} 1 & -1 & 3 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 3 & -7 & 2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & -1 & 3 & 0 \\ 0 & 2 & -5 & 1 \\ 1 & 3 & -7 & 2 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & -1 & 3 & 0 \\ 0 & 2 & -5 & 1 \\ 0 & 4 & -10 & 2 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & -1 & 3 & 0 \\ 0 & 2 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

$\begin{pmatrix} 1 & -1 & 3 & 0 \\ 0 & 2 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a row echelon form of A.

(3 pts for applying row operations correctly. Students get 1 point deduction for minor mistakes in row operations.
1 pt for a correct row echelon form of A. Please note that there are various REFs.)

(b) Nonzero row vectors of a REF consist a basis of the row space. Hence $\{(1, -1, 3, 0), (0, 2, -5, 1)\}$ is a basis of the row space of A. The rank of A is the dimension of the row space which is 2.
(1 pt for choosing nonzero row vectors as a basis. 1 pt for the rank of A. If students have wrong REF in (a) but answer (b) with correct reasoning, they get 1 pt.)

(c) Since $\mathbf{v}$ belongs to the row space, we have

$$\mathbf{v} = (a, 2, b, -1) = x(1, -1, 3, 0) + y(0, 2, -5, 1), \quad \text{for some constants } x, y.$$

Thus $2 = -x + 2y,\ -1 = y$. We can solve that $y = -1$ and $x = -4$.

$$\mathbf{v} = (a, 2, b, -1) = -4(1, -1, 3, 0) - (0, 2, -5, 1) = (-4, 2, -7, -1).$$

Hence $a = -4,\ b = -7$.
(1 pt for writing $\mathbf{v}$ as a linear combination of basis vectors. 1 pt for solving the coefficients of the linear combination. 2 points for the final answer $a = -4,\ b = -7$.)

2. (10%) A is a $n \times n$ symmetric matrix. Mark "O" for correct statements and "X" for false statements.

(2 pts for each answer.)

(a) __**O**__ If $\mathbf{v}$ is an eigenvector of A, then $\mathbf{v}$ is an eigenvector of A^n for all positive integers n.

(b) __**O**__ If $\mathbf{v}_1,\ \mathbf{v}_2$ are eigenvectors of A with respect to different eigenvalues, then $\mathbf{v}_1,\ \mathbf{v}_2$ are orthogonal.

(c) __**X**__ Suppose that B is a REF of A. Then B and A have same eigenvalues.

(d) __**X**__ A^{2k} is positive definite for all positive integers k.

(e) __**O**__ If A is negative definite, then $-A$ is positive definite.

3. Following the steps to find the maximum value of $f(x,y) = \min\{x, 2y\}$ under the constraint $4x^2 + 4xy + y^2 = 4$. (It is already known that the maximum value exists.)

(a) (7%) Solve the optimization problem: (Don't forget to verify NDCQ.)

$$\text{Maximize } x \text{ subject to } x \le 2y,\ 4x^2 + 4xy + y^2 = 4.$$

(b) (7%) Solve the optimization problem: (Don't forget to verify NDCQ.)

$$\text{Maximize } 2y \text{ subject to } 2y \le x,\ 4x^2 + 4xy + y^2 = 4.$$

(c) (2%) Find the maximum value of $f(x,y) = \min\{x, 2y\}$ on the constraint set $4x^2 + 4xy + y^2 = 4$.

Solution:

(a) We first check NDCQ. (You need to **verify** NDCQ, instead of just claiming that NDCQ is satisfied.) Suppose that $x - 2y \le 0$ is binding. We consider the Jacobian matrix

$$\begin{pmatrix} 8x+4y & 4x+2y \\ 1 & -2 \end{pmatrix}.$$

It is of full rank 2 unless $2x + y = 0$. However, the given constraint is $(2x+y)^2 = 4$. Thus $2x + y \ne 0$ and hence the Jacobian is of full rank.

Suppose next that $x - 2y < 0$. Note that the Jacobian matrix

$$\begin{pmatrix} 8x+4y & 4x+2y \end{pmatrix}.$$

The Jacobian matrix is not of full rank only if $2x + y = 0$. Then $4x^2 + 4xy + y^2 = (2x+y)^2 = 0$, which violates the constraint. Thus, we see that the Jacobian is of full rank. (2%, 1% for $x - 2y \le 0$ binding, 1% for $x - 2y < 0$)

Then we consider Lagrangian

$$L = x - \mu(4x^2 + 4xy + y^2 - 4) - \lambda(x - 2y). \quad (1\%)$$

The FOC are

$$\begin{array}{ll} L_x = 1 - (8x+4y)\mu - \lambda = 0 & (1) \\ L_y = -(4x+2y)\mu - (-2)\lambda = 0 & (2) \\ L_\mu = 4x^2 + 4xy + y^2 - 4 = 0 & (3) \\ \lambda(x - 2y) = 0 & (4) \\ x - 2y \le 0 & (5) \\ \lambda \ge 0 & (6) \end{array} \quad (2\%)$$

There are two solutions, $(x, y, \mu, \lambda) = (\frac{4}{5}, \frac{2}{5}, \frac{1}{10}, \frac{1}{5})$ and $(x, y, \mu, \lambda) = (-\frac{4}{5}, -\frac{2}{5}, -\frac{1}{10}, \frac{1}{5})$. (2%, 1% for each)

For each excess solution, -1%, at most -2%.

(b) We first check NDCQ. Suppose that $2y - x \le 0$ is binding. We consider the Jacobian matrix

$$\begin{pmatrix} 8x+4y & 4x+2y \\ -1 & 2 \end{pmatrix}.$$

It is of full rank 2 unless $2x + y = 0$. However, the given constraint is $(2x+y)^2 = 4$. Thus $2x + y \ne 0$ and hence the Jacobian is of full rank.

Suppose that $2y - x \le 0$ is binding. Then $25y^2 = 4$, hence $y = \pm\frac{2}{5}$ and $x = \pm\frac{4}{5}$. Thus $f(\pm\frac{4}{5}, \pm\frac{2}{5}) = \pm\frac{4}{5}$.

Suppose next that $2y - x < 0$. Note that the Jacobian matrix

$$\begin{pmatrix} 8x+4y & 4x+2y \end{pmatrix}.$$

The Jacobian matrix is not of full rank only if $2x + y = 0$. Then $4x^2 + 4xy + y^2 = (2x+y)^2 = 0$, which violates the constraint. Thus, we see that the Jacobian is of full rank. (2%)

Then we consider Lagrangian

$$L = 2y - \mu(4x^2 + 4xy + y^2 - 4) - \lambda(2y - x). \quad (1\%)$$

The FOC are

$$\begin{aligned} &L_x = -(8x+4y)\mu - (-1)\lambda = 0 && (1)\\ &L_y = 2-(4x+2y)\mu - 2\lambda = 0 && (2)\\ &L_\mu = 4x^2+4xy+y^2-4 = 0 && (3)\\ &\lambda(2y-x) = 0 && (4)\\ &2y-x \le 0 && (5)\\ &\lambda \ge 0 && (6) \end{aligned} \quad (2\%)$$

There are two solutions, $(x,y,\mu,\lambda) = (\frac{4}{5}, \frac{2}{5}, \frac{1}{10}, \frac{8}{10})$ and $(x,y,\mu,\lambda) = (-\frac{4}{5}, -\frac{2}{5}, -\frac{1}{10}, \frac{8}{10})$. (2%, 1% for each)

For each excess solution, -1%, at most -2%

(c) We have maximizer $(x,y) = (\frac{4}{5}, \frac{2}{5})$ in both cases. Hence $f_{max} = \frac{4}{5}$. (2%)

4. Consider the problem :

$$\text{Maximize } u(x,y) = -e^{-2x} - e^{-3y} \text{ subject to the constraints } 4x + y \le 10,\ x \ge 0,\ y \ge 0.$$

(a) (2%) Verify that Kuhn-Tucker's NDCQ is valid.

(b) (2%) Write down the Kuhn-Tucker's Lagrangian function.

(c) (4%) Write down the Kuhn-Tucker's first order conditions.

(d) (2%) Explain why $4x + y \le 10$ is binding at any solution to first order conditions.

(e) (8%) Solve the optimization problem.

Solution:

(a) Suppose $4x + y \le 10$ is binding.

- If $x = 0$, then $y = 10$ and the reduced Jacobian matrix is (1) which is clearly of rank 1.
- If $y = 0$, then $x = 2.5$ and the reduced Jacobian matrix is (4) which is clearly of rank 1.
- If both $x, y \neq 0$, then we have the full Jacobian matrix $(4, 1)$ which is of rank 1.

In all possible cases, the (reduced) Jacobian matrix has full rank so Kuhn-Tucker's NDCQ is verified.
Grading Scheme.

- (0.5%+0.5%+0.5%) For listing all possible reduced Jacobian matrices
- (0.5%) Mention that all of them has full rank (or of rank 1).

(b) $\widetilde{L}(x,y,\lambda) = -e^{-2x} - e^{-3y} - \lambda(4x + y - 10)$.
Grading Scheme.

- All or nothing.

(c) Grading Scheme.
(2%)

$$x(2e^{-2x} - 4\lambda) = 0 \qquad (1)$$
$$y(3e^{-3y} - \lambda) = 0 \qquad (2)$$
$$\lambda(4x + y - 10) = 0 \qquad (3)$$

(1%)

$$2e^{-2x} - 4\lambda \le 0 \qquad (4)$$
$$3e^{-3y} - \lambda \le 0 \qquad (5)$$

(1%)

$$4x + y \le 10, x \ge 0, y \ge 0 \qquad (6)$$
$$\lambda \ge 0 \qquad (7)$$

Remark. -0.5% for each calculation mistake.

(d) By (4), $\lambda \ge 2e^{-2x} > 0$ so (3) implies $4x + y = 10$.
Grading Scheme.

- (1%) Noticing that (4) or (5) implies λ is strictly positive.
- (1%) Overall coherency and quality of the argument.

(e)
- (2%) If $x = 0$, then $y = 10$. By (2), $\lambda = 3e^{-30}$, which violates (4) that $\lambda \ge \frac{1}{2}$.

- (2%) If $y = 0$, then $x = 2.5$. By (1), $\lambda = e^{-5}/2$. This violates (4) that $\lambda \geq 3$.

Therefore, we must have $xy \neq 0$. (1) and (2) thus become $2e^{-2x} = 4\lambda$ and $3e^{-3y} = \lambda$.

Hence,

$$\underbrace{e^{-2x} = 6e^{-3y}}_{2\%} \Rightarrow -2x = \ln 6 - 3y.$$

Together with $4x + y = 10$, we obtain

$$(2\%) \quad x = \frac{10 + 2\ln 6}{7},\ y = \frac{30 - \ln 6}{14},\ \lambda = \frac{1}{2}e^{-(20+4\ln 6)/7}$$

Since this is the only solution to the Kuhn-Tucker's FOC, it must be the maximizer.

There are four major components of the grading scheme :

- (2%) Explain, with a correct argument, why x must be non-zero (or equivalently prove that when $x = 0$, $y = 0$; $x = 0$, $y \neq 0$ lead to no solutions)
- (2%) Explain, with a correct argument, why y must be non-zero (or equivalently prove that when $y = 0$, $x = 0$; $y = 0$, $x \neq 0$ lead to no solutions)
- (2%) In the case $xy \neq 0$, derive the correct relation between x and y
- (2%) Solving for the correct maximizer.

Depending on the quality and/or accuracy of writing, marks may be taken away from each part.

5. Suppose that $(x, y, z, \mu_1, \mu_2) = (1, \sqrt{2}, 1, 1, 0)$ is a maximizer to the following optimization problem: Maximize $f(x, y, z) = xy^2 z$ subject to $h_1(x, y, z) = x^2 + y^2 + z^2 = 4$ and $h_2(x, y, z) = x + y^2 + z = 4$.

   (a) (2%) Show that NDCQ is satisfied at $(x, y, z) = (1, \sqrt{2}, 1)$.

   (b) (6%) Estimate the maximum value of $xy^2 z + 0.1y^2$ subject to $x^2 + y^2 + z^2 = 4.2$ and $x + y^2 + z = 4.1$.

Solution:

(a) We consider the Jacobian matrix

$$\begin{pmatrix} 2x & 2y & 2z \\ 1 & 2y & 1 \end{pmatrix}$$

At the point $(1, \sqrt{2}, 1, 1, 0)$,

$$\begin{pmatrix} 2 & 2\sqrt{2} & 2 \\ 1 & 2\sqrt{2} & 1 \end{pmatrix}$$

has rank 2. So NDCQ is satisfied. (2 %).

(b) We consider the following optimization problem: Maximize $xy^2 z + a_1 y^2$ subjects to $x^2 + y^2 + z^2 = a_2$ and $x + y^2 + z = a_3$. (2%)

The Lagrangian is

$$L = xy^2 z + a_1 y^2 - \mu_1(x^2 + y^2 + z^2 - a_2) - \mu_2(x + y^2 + z - a_3). \quad (1\%)$$

When $(a_1, a_2, a_3) = (0, 4, 4)$, we have that maximum value $f(1, \sqrt{2}, 1) = 2$. (1%)

By Envelope Theorem, we have

$$\frac{\partial f_{max}}{\partial a_1} = \frac{\partial L}{\partial a_1}|_{\mathbf{p}} = y^2|_{\mathbf{p}} = 2.$$

$$\frac{\partial f_{max}}{\partial a_2} = \frac{\partial L}{\partial a_2}|_{\mathbf{p}} = \mu_1|_{\mathbf{p}} = 1.$$

$$\frac{\partial f_{max}}{\partial a_3} = \frac{\partial L}{\partial a_3}|_{\mathbf{p}} = \mu_2|_{\mathbf{p}} = 0.$$

Thus $f_{max}(0.1, 4.2, 4.1) \approx f_{max}(0, 4, 4) + 0.1 \cdot 2 + 0.2 \cdot 1 + 0.1 \cdot 0 = 2.4$. (2%)

6. Consider the following quadratic form

$$f(x,y,z) = x^2 - 2y^2 + 2xy + 2xz - 2yz.$$

(a) (3%) Write down the symmetric matrix A associated with the above quadratic form.

(b) (6%) Compute all the leading principal minors (LPM) of the matrix A. Hence, determine whether $(0,0,0)$ is a local maximum, local minimum or saddle point for f.

(c) Now subject $f(x,y,z)$ to the constraint $2y + z = 0$.

(i) (1%) Find the value of μ^* such that $(0,0,0,\mu^*)$ is a critical point of $L(x,y,z,\mu)$.

(ii) (5%) Write down the bordered Hessian matrix at $(0,0,0,\mu^*)$.

(iii) (5%) Use the second order condition to determine whether $(0,0,0)$ is a local maximum, local minimum or saddle point when f is being constrained.

Solution:

(a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$.

Grading Scheme :
-0.5% for each incorrect entry.

(b) $\text{LPM}_1 = 1$, $\text{LPM}_2 = -3$, $\text{LPM}_3 = -1$.

As $(-1)\text{LPM}_1 < 0$ and $\text{LPM}_2 < 0$, the Sylvester's criterion implies that A is indefinite. Hence, $(0,0,0)$ is a saddle point for q.

Grading Scheme :
(1%) For correct LPM_1 and LPM_2
(2%) For correct LPM_3
(2%) For correct use of Sylvester's criterion :note that a candidate needs to specify indices i, j such that $\text{LPM}_i < 0$ and $(-1)^j\text{LPM}_j < 0$; just saying that 'LPM does not match the sign pattern' without specifying which/ what 'sign parttern' is considered as incomplete.)
(1%) Mentioning indefinite and hence saddle point.

(c) (i) $\mu^* = 0$ (1%: All or nothing)

(ii) Let $L(x,y,z,\mu) = f(x,y,z) - \mu(2y+z)$ be the Lagrangian function. Let $h(x,y,z) = 2y + z$.
The bordered Hessian matrix (at $(0,0,0)$) is

$$\begin{pmatrix} 0 & h_x & h_y & h_z \\ h_x & L_{xx} & L_{xy} & L_{xz} \\ h_y & L_{xy} & L_{yy} & L_{yz} \\ h_z & L_{xz} & L_{zy} & L_{zz} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 2 & 2 \\ 2 & 2 & -4 & -2 \\ 1 & 2 & -2 & 0 \end{pmatrix}.$$

Grading Scheme :
-1% for each incorrect entry.
No marks for students whose matrix is not even 4×4.
We accept the answer $\begin{pmatrix} 0 & 0 & -2 & -1 \\ 0 & 2 & 2 & 2 \\ -2 & 2 & -4 & -2 \\ -1 & 2 & -2 & 0 \end{pmatrix}$ (the 'border' gets negafied).

(iii) (1%) Since there are 3 variables and 1 constraint, we need to check the last 2 LPMs. As
(0.5%) $\text{LPM}_3 = 2(-2) = -4$
(0.5%) $\text{LPM}_4 = -1$.
(1%) Since the LPMs have the same signs and moreover
(1%) LPM_4 has the same sign as $(-1)^1$,
(1%) the second order condition implies that $(0,0,0)$ a local minimum.

Grading Scheme :
(1%) Knowing how many LPMs need to be checked
(0.5+0.5%) Correct values of the last two LPM

(1%) Checking hypothesis of SOC about same/alternating signs
(1%) Checking hypothesis of SOC about matching the signs of the largest LPM with $(-1)^m$ or $(-1)^n$,
(1%) Correct conclusion

Remark : a student gets at most 3% (for essentially knowing the statement of SOC) if their bordered Hessian matrix was incorrect.

7. Consider the following optimization problem:

(A) Maximize $f(x,y,z) = x + 2y + z^2$ subject to constraints $x^2 + y^2 + z^2 = 2$ and $y = 1$.

Alex observes that he can plug in $y = 1$ and eliminate the variable y. Hence he solves another optimization problem:

(B) Maximize $F(x,z) = x + 2 + z^2$ subject to the constraint $x^2 + z^2 = 1$.

Cathy thinks this is a clever way to simplify the problem but she wants to carefully check that second order conditions (SOC) for problem (A) and (B) derive the same result.

(a) (4%) Write down Lagrangian functions for problem (A) and (B) which we denote by $L_A(x,y,z,\mu_1,\mu_2)$ and $L_B(x,z,\mu_1)$.

(b) (5%) Cathy has shown that if $(x^*,1,z^*,\mu_1^*,\mu_2^*)$ is a solution to the FOC for problem (A), then (x^*,z^*,μ_1^*) is a solution to the FOC for problem (B).
Now write down the bordered Hessian matrix, H_A, for problem (A) at $(x^*,1,z^*,\mu_1^*,\mu_2^*)$, and the bordered Hessian matrix, H_B, for problem (B) at (x^*,z^*,μ_1^*).

(c) (2%) Find the constant c such that

$$\det H_A(x^*,1,z^*,\mu_1^*,\mu_2^*) = c \cdot \det H_B(x^*,z^*,\mu_1^*).$$

(d) (7%) Describe the second order conditions for problem (A) and (B) by filling out the table.

Optimization Problem	(A)	(B)
number of variables (n)	3	2
number of constraints (m)	2	1
SOC for local maximum	Check the last 1 LPM(s) of H_A. It/They should satisfy $\det H_A < 0$ (i.e. $(-1)^n \det H_A > 0$)	Check the last 1 LPM(s) of H_B. It/They should satisfy $\det H_B > 0$ (i.e. $(-1)^n \det H_B > 0$)
SOC for local minimum	Check the last 1 LPM(s) of H_A. It/They should satisfy $\det H_A > 0$ (i.e. $(-1)^m \det H_A > 0$)	Check the last 1 LPM(s) of H_B. It/They should satisfy $\det H_B < 0$ (i.e. $(-1)^m \det H_B > 0$)

Show that SOC for problem (A) and (B) derive the same result.

Solution:

(a)

$$L_A(x,y,z,\mu_1,\mu_2) = x + 2y + z^2 - \mu_1(x^2+y^2+z^2-2) - \mu_2(y-1).$$
$$L_B(x,z,\mu_1) = (x + 2 + z^2) - \mu_1(x^2+z^2-1).$$

(2 pts for L_A, 2 pts for L_B.)

(b)

$$H_A = \begin{pmatrix} 0 & 0 & 2x^* & 2 & 2z^* \\ 0 & 0 & 0 & 1 & 0 \\ 2x^* & 0 & -2\mu_1^* & 0 & 0 \\ 2 & 1 & 0 & -2\mu_1^* & 0 \\ 2z^* & 0 & 0 & 0 & 2-2\mu_1^* \end{pmatrix}$$

$$H_B = \begin{pmatrix} 0 & 2x^* & 2z^* \\ 2x^* & -2\mu_1^* & 0 \\ 2z^* & 0 & 2-2\mu_1^* \end{pmatrix}$$

(3 pts for H_A. 2 pts for H_B. Students get 1 point deduction or 2 points deduction for minor mistakes.)

(c) To compute $\det H_A$, we can expand the determinate with respect to the second row. Then we expand the determinate with respect to the second column. Thus

$$\det H_A = 1 \cdot \det \begin{pmatrix} 0 & 0 & 2x^* & 2z^* \\ 2x^* & 0 & -2\mu_1^* & 0 \\ 2 & 1 & 0 & 0 \\ 2z^* & 0 & 0 & 2-2\mu_1^* \end{pmatrix} = (-1) \cdot \det \begin{pmatrix} 0 & 2x^* & 2z^* \\ 2x^* & -2\mu_1^* & 0 \\ 2z^* & 0 & 2-2\mu_1^* \end{pmatrix} = -\det H_B.$$

Hence $\det H_A = c \cdot \det H_B$ where $c = -1$.
(1 pt for expanding $\det H_A$ with respect to the second row and the second column. 1 pt for $c = -1$.)

(d) SOCs for problem (A) and (B) are listed in the table. Because $\det H_A = -\det H_B$ and SOCs for local maximum and local minimum require different signs of $\det H_A$ and $\det H_B$, we conclude that SOCs derive the same result.
(0.5 pt for each n, m in the table. 1 pt for each SOC in the table. 1 pt for arguing that SOCs derive the same result.)