1112 模組01-05班 微積分4 期考解答和評分標準

1. Consider the vector field $\mathbf{F}(x,y) = \dfrac{2(x-y)}{x^2+y^2}\,\mathbf{i} + \dfrac{2(x+y)}{x^2+y^2}\,\mathbf{j}$

   (a) (4%) Evaluate, directly, $\oint_{C_r} \mathbf{F}\cdot d\mathbf{r}$, where C_r is the circle $x^2+y^2=r^2$, $r>0$, oriented counterclockwise.

   (b) (6%) Determine whether $\mathbf{F}$ is conservative on each of the following regions.

   (i) $\{(x,y)\in\mathbb{R}^2 : 1 < x^2+y^2 < 4\}$

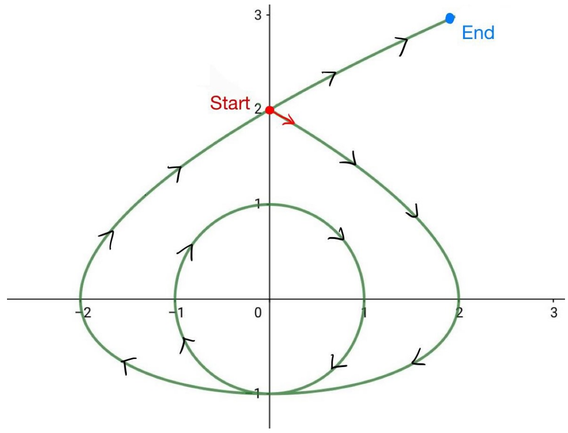
   (ii) $\{(x,y)\in\mathbb{R}^2 : y > 0\}$

   In the case if $\mathbf{F}$ is conservative, find its scalar potential function.

   (c) (6%) Let C be the portion of the curve

   $$(x^2-y^3+3y^2-4)(x^2+y^2-1)=0$$

   that begins at $(0,2)$, winding the origin twice *clockwise* and ends at $(2,3)$ (see figure below).
   Using (a) and (b), evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$.



Solution:

(a) (1M) Parametrize C by $\mathbf{r}(t) = \langle R\cos t, R\sin t\rangle$, $\quad 0\le t\le 2\pi$, then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \underbrace{\left\langle \frac{2(\cos t-\sin t)}{R}, \frac{2(\cos t+\sin t)}{R}\right\rangle}_{(1M)} \cdot \underbrace{\langle -R\sin t, R\cos t\rangle}_{(1M)}, dt$$
$$= \int_0^{2\pi} -2\cos t\sin t + 2\sin^2 t + 2\cos^2 t + 2\cos t\sin t\, dt$$
$$= \underbrace{4\pi}_{(1M)}$$

<u>Grading scheme for 1a.</u>

- (1M) Correct parametrization for C
- (1M) Definition of Line integral ($\vec{F}(\vec{r}(t))$)
- (1M) Definition of Line integral ($\vec{r}\,'(t)$)
- (1M) ***Correct answer

<u>Remarks.</u>

(a) At most 1M will be deducted overall if a student messed up the orientation of C (and lead to a sign error)

(b) (i) (1M) Take $R = 1.5$ in (a) which is a curve inside the region for which $\oint_{C_R} \mathbf{F}\cdot d\mathbf{r} = 4\pi \neq 0$.
(1M) Therefore, $\mathbf{F}$ is not conservative on this region.

(ii) (1M) $\dfrac{\partial Q}{\partial x} = \dfrac{2(x^2+y^2)-2(x+y)(2x)}{(x^2+y^2)^2} = \underline{\dfrac{2y^2-4xy-2x^2}{(x^2+y^2)^2}}$

$\dfrac{\partial P}{\partial y} = \dfrac{-2(x^2+y^2)-2(x-y)(2y)}{(x^2+y^2)^2} = \dfrac{2y^2-4xy-2x^2}{(x^2+y^2)^2}$

Since $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$ and the given region is (1M) simply-connected, we can conclude that **F** is conservative on this region. To find the scalar potential, we set

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y^2} - \dfrac{2y}{x^2+y^2} \\ \dfrac{\partial f}{\partial y} = \dfrac{2x}{x^2+y^2} + \dfrac{2y}{x^2+y^2} \end{cases} \Longrightarrow \begin{cases} f = \ln(x^2+y^2) - 2\tan^{-1}\dfrac{x}{y} + A(y) \\ f = -2\tan^{-1}\dfrac{x}{y} + \ln(x^2+y^2) + B(x) \end{cases}$$

So we can take (2M) $f(x,y) = -2\tan^{-1}\dfrac{x}{y} + \ln(x^2+y^2) + C$

Grading scheme for 1b.

- (1M) Correct argument for (i)
- (1M) Correct conclusion for (i)
- (1M) Calculating, explicitly, either Q_x or P_y for (ii)
- (1M) Mentioning D is 'simply-connected' for (ii)
- (2M) Correct potential function (can omit '+C')

Remarks.

(a) At most 1M will be deducted overall if a student messed up the orientation of C (and lead to a sign error)

(c) **Claim.** Any anti-clockwisely oriented simple closed curve **C** that encloses origin satisfies

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 4\pi$$

Proof. (3%) Let C' be a circle small enough to be fitted inside the curve.(anti-clockwise oriented) and D be the region bounded by C and C'.

As **F** is C^1 on D, Generalized Green's Theorem implies that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} - \oint_{C'} \mathbf{F}\cdot d\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = 0 \Longrightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_{C'} \mathbf{F}\cdot d\mathbf{r} = 4\pi$$

(1%) Decompose C into two clockwise oriented closed curves C_1, C_2, a segment C_3 start from $(0,2)$ to $(2,3)$.

- By Claim $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = -4\pi$
- (1%) By FTC, $\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = f(2,3) - f(0,2) = \ln 13 - 2\tan^{-1}\dfrac{2}{3} - \ln 4 = \ln\dfrac{13}{4} - 2\tan^{-1}\dfrac{2}{3}$

So $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_{C_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \underbrace{\ln\dfrac{13}{4} - 2\tan^{-1}\dfrac{2}{3} - 8\pi}_{(1M)}$

Grading scheme for 1c.

- (3M) Correct and complete argument to prove the 'claim' via Generalized Green's Theorem
- (1M) Decomposing the given non-simple curve into three simple pieces.
- (1M) For using FTC to calculate the line integral along C_3.
- (1M) Overall correct answer.

2. Suppose that $f(x,y)$ is a scalar function that has continuous second order partial derivatives for $(x,y) \in \mathbb{R}^2$. For $r > 0$, let C_r be the circle $x^2 + y^2 = r^2$ parametrized by $\mathbf{r}(t) = \langle r\cos(t),\, r\sin(t)\rangle$, $0 \le t \le 2\pi$.

(a) (4%) Let $A(r) = \dfrac{1}{2\pi r}\oint_{C_r} f(x,y)\,\mathrm{d}s$ be the 'average value' of f on the circle C_r. By the parametrization $\mathbf{r}(t)$, write $A(r)$ as a definite integral with respect to t. Hence, find a function $g(r,t)$ such that

$$A(r) = \int_0^{2\pi} g(r,t)\,\mathrm{d}t.$$

(b) (5%) Find the functions $P(x,y)$ and $Q(x,y)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}r}A(r) = \frac{1}{r}\oint_{C_r} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y.$$

Express your answers in terms of the first order partial derivatives of $f(x,y)$.
(**Hint.** You may use, without proof, the fact that $\dfrac{\mathrm{d}}{\mathrm{d}r}A(r) = \displaystyle\int_0^{2\pi}\frac{\partial}{\partial r}g(r,t)\,\mathrm{d}t$.)

(c) (2%) Use (b) and Green's Theorem to find a function $R(x,y)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}r}A(r) = \frac{1}{r}\iint_{D_r} R(x,y)\,\mathrm{d}A,$$

where D_r is the disc $x^2 + y^2 \le r^2$. Express your answer in terms of the second order partial derivatives of f.

(d) (2%) We can show that $\lim_{r\to 0^+} A(r) = f(0,0)$. Suppose that $f_{xx} + f_{yy} = 1$ and $f(0,0) = 0$. Find $A(r)$.

Solution:

(a) Note that $\mathbf{r}'(t) = (-r\sin t, r\cos t)$ and $|\mathbf{r}'(t)| = r$. Thus

$$A(r) = \frac{1}{2\pi r}\int_0^{2\pi} f(r\cos t, r\sin t)|\mathbf{r}'(t)|\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} f(r\cos t, r\sin t)\,\mathrm{d}t \quad \text{(3 points)}.$$

Suppose that $A(r) = \displaystyle\int_0^{2\pi} g(r,t)\,\mathrm{d}t$. Then $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} f(r\cos t, r\sin t)\,\mathrm{d}t = \int_0^{2\pi} g(r,t)\,\mathrm{d}t$.
Hence $g(r,t) = \dfrac{1}{2\pi}f(r\cos t, r\sin t)$ (1 point)

(b)

$$\begin{aligned}\frac{d}{dr}A(r) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{d}{dr}(f(r\cos t, r\sin t))\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} f_x(r\cos t, r\sin t)\cos t + f_y(r\cos t, r\sin t)\sin t\,\mathrm{d}t. \quad \text{(2 points)}\end{aligned}$$

Suppose that $\dfrac{d}{dr}A(r) = \dfrac{1}{r}\displaystyle\int_{C_r} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$.

$$\begin{aligned}\text{Then } \frac{d}{dr}A(r) &= \frac{1}{r}\int_0^{2\pi} P(r\cos t, r\sin t)(-r\sin t) + Q(r\cos t, r\sin t)r\cos t\,\mathrm{d}t \\ &= \int_0^{2\pi} -P(r\cos t, r\sin t)\sin t + Q(r\cos t, r\sin t)\cos t\,\mathrm{d}t \quad \text{(2 points)}\end{aligned}$$

$$\begin{aligned}\because\ &\frac{1}{2\pi}\int_0^{2\pi} f_x(r\cos t, r\sin t)\cos t + f_y(r\cos t, r\sin t)\sin t\,\mathrm{d}t \\ &= \int_0^{2\pi} -P(r\cos t, r\sin t)\sin t + Q(r\cos t, r\sin t)\cos t\,\mathrm{d}t \\ \therefore\ & P = -\frac{1}{2\pi}f_y,\ Q = \frac{1}{2\pi}f_x \quad \text{(1 point)}\end{aligned}$$

(c) $\dfrac{d}{dr}A(r) = \dfrac{1}{r}\displaystyle\oint_{C_r} P\,\mathrm{d}x + Q\,\mathrm{d}y = \frac{1}{r}\iint_{D_r}\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A = \frac{1}{r}\iint_{D_r}\frac{1}{2\pi}(f_{xx} + f_{yy})\,\mathrm{d}A$ (1 point)
Hence $R(x,y) = \dfrac{1}{2\pi}(f_{xx} + f_{yy})$ (1 point)

(d) Because $f_{xx} + f_{yy} = 1$,

$$\frac{d}{dr}A(r) = \frac{1}{r}\iint_{D_r} \frac{1}{2\pi}(f_{xx} + f_{yy})\,\mathrm{d}A = \frac{1}{r}\iint_{D_r} \frac{1}{2\pi}\,\mathrm{d}A = \frac{r}{2}. \quad \text{1 point}$$

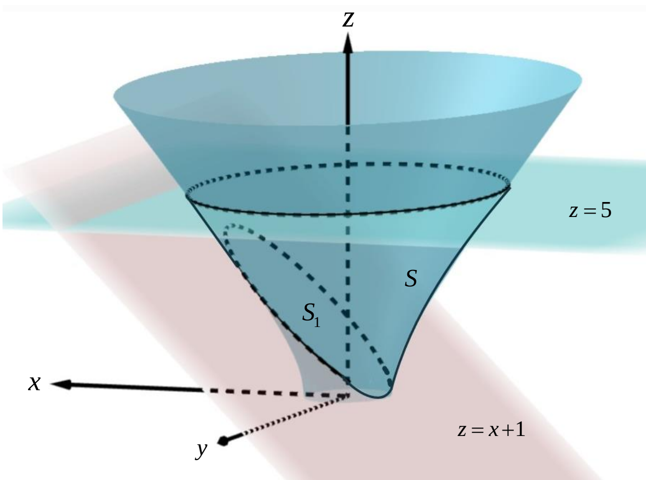
Thus $A(r) = \dfrac{r^2}{4} + c$. $\because \lim_{r \to 0^+} A(r) = 0 \therefore c = 0$ and $A(r) = \dfrac{r^2}{4}$. (1 point)

3. In the figure below,

- S_1 is part of the plane $z = x + 1$ satisfying $2x^2 + y^2 - z^2 \le 2$;
- S is part of the surface $2x^2 + y^2 - z^2 = 2$ between the planes $z = x + 1$ and $z = 5$.

Both surfaces are endowed with downward orientation. Consider the vector field

$$\mathbf{F}(x, y, z) = (e^x - z)\,\mathbf{i} + (e^y + x)\,\mathbf{j} + e^z\,\mathbf{k}.$$



(a) (8%) Parametrize S_1 and thus compute $\displaystyle\iint_{S_1} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

(b) (6%) Evaluate $\displaystyle\iint_{S \cup S_1} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

Solution:

(a) Find the intersection of $z = x + 1$ and $2x^2 + y^2 - z^2 = 2$. $\Rightarrow 2x^2 + y^2 - (x+1)^2 = 2 \Rightarrow (x-1)^2 + y^2 = 4$.
Hence the projection of S_1 onto the xy-plane is $D = \{(x, y) | (x-1)^2 + y^2 \le 4\}$.

Solution 1:
One parametrization of S_1 is $\mathbf{r}(x, y) = (x, y, x+1)$, $(x, y) \in D$. (1 point for $\mathbf{r}(x, y)$. 2 points for D.)
$\mathbf{r}_x \times \mathbf{r}_y = (-1, 0, 1)$ which is upward and is in the opposite direction of the normal vector. (1 point)

Moreover, $\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x - z & e^y + x & e^z \end{vmatrix} = (0, -1, 1)$ (2 points).

$$\text{Hence } \iint_{S_1} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_D (0, -1, 1) \cdot (-\mathbf{r}_x \times \mathbf{r}_y)\, dx\, dy \quad \text{(1 point)}$$
$$= \iint_D -1\, dA = -A(D) = -4\pi \quad \text{(1 point)}$$

Solution 2:
Another parametrization of S_1 is $\mathbf{r}(r, \theta) = (1 + r\cos\theta, r\sin\theta, 2 + r\cos\theta)$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$.
(2 points for $\mathbf{r}(r, \theta)$, 1 point for ranges of r and θ)
$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & \cos\theta \\ -r\sin\theta & r\cos\theta & -r\sin\theta \end{vmatrix} = (-r, 0, r)$ which is upward and is in the opposite direction of the normal vector. (1 point). $\operatorname{curl}(\mathbf{F}) = (0, -1, 1)$ (2 points)

$$\text{Hence } \iint_{S_1} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^2 (0, -1, 1) \cdot (-\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta \quad \text{(1 point)}$$
$$= \int_0^{2\pi} \int_0^2 -r\, dr\, d\theta = -4\pi \quad \text{(1 point)}$$

Solution 3:
By Stokes' Theorem, we know that $\displaystyle\iint_{S_1} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S_1} \mathbf{F} \cdot d\mathbf{r}$. (1 point)
$\because S_1$ has downward orientation $\therefore \partial S_1$ is oriented clockwise.

Parametrize ∂S_1 by

$$\mathbf{r}(t) = (1 + 2\cos t, -2\sin t, 2 + 2\cos t),\ 0 \le t \le 2\pi.\ \text{(1 point)}.$$

$$\int_{\partial S_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\mathrm{t}))\cdot \mathbf{r}'(\mathrm{t})\,\mathrm{dt}\quad \text{(1 point)}$$

$$= \int_0^{2\pi} -2e^{1+2\cos t}\sin t + 4\sin t + 4\sin t\cos t - 2e^{-2\sin t}\cos t - 2\cos t - 4\cos^2 t - 2e^{2+2\cos t}\sin t\,dt$$

$$= -4\pi.\quad \text{(2 points)}$$

(Computing $\iint_{S_1} \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S}$ by Stokes' Theorem can get at most 5 points because it doesn't provide a parametrization of S_1.)

(b) **Solution 1:** Let S_2 be the part of the plane $z = 5$ satisfying $2x^2 + y^2 \le 27$ with downward orientation. Then $S \cup S_1$ and S_2 have the same oriented boundary curve. Hence by Stokes' Theorem,

$$\iint_{S\cup S_1} \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_{\mathrm{S_2}} \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S}\quad \text{(2 points)}$$

Because the unit normal vector of S_2 is $(0, 0, -1)$,

$$\iint_{S_2} \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_{S_2} \mathrm{curl}(\mathbf{F})\cdot(0,0,-1)\mathrm{dS} = \iint_{\mathrm{S_2}} -1\mathrm{dS}\quad \text{(2 points)}$$
$$= -A(S_2).$$

Since S_2 is an ellipse, the area of S_2 is $\pi\cdot\sqrt{27}\cdot\sqrt{\frac{27}{2}} = \frac{27\pi}{\sqrt{2}}$.

Hence $\iint_{S\cup S_1} \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_{\mathrm{S_2}} \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S} = -\mathrm{A}(\mathrm{S_2}) = -\frac{27\pi}{\sqrt{2}}$ (2 points)

Solution 2: $\partial(S \cup S_1)$ is the curve C with parametrization $\mathbf{r}(t) = (3\sqrt{\frac{3}{2}}\cos t, -3\sqrt{3}\sin t, 5)$, $0 \le t \le 2\pi$. (2 points).

By Stokes' Theorem,

$$\iint_{S\cup S_1} \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S} = \int_{\mathrm{C}} \mathbf{F}\cdot d\mathbf{r}\quad \text{(1 point)}$$

$$= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\mathbf{t}))\cdot \mathbf{r}'(\mathbf{t})\mathbf{dt}$$

$$= \int_0^{2\pi} -(e^{3\sqrt{\frac{3}{2}}\cos t} - 5)3\sqrt{\frac{3}{2}}\sin t - (e^{-3\sqrt{3}\sin t} + 3\sqrt{\frac{3}{2}}\cos t)3\sqrt{3}\cos t\,dt\quad \text{(1 point)}$$

$$= -\frac{27}{\sqrt{2}}\pi\quad \text{(2 points)}$$

4. Let $f(x,y,z)$ be a scalar field and $\mathbf{G}(x,y,z)$ be a vector field, both smooth (that is, partial derivatives exist in any order). Let D be a solid region in $\mathbb{R}^3$ with boundary surface ∂D oriented outward.

(a) (4%) Prove that $\operatorname{div}(f\mathbf{G}) = \nabla f \cdot \mathbf{G} + f\operatorname{div}(\mathbf{G})$.

(b) (2%) Prove that $\iiint_D \nabla f \cdot \mathbf{G}\, \mathrm{d}V = \iint_{\partial D} f\mathbf{G} \cdot \mathrm{d}\mathbf{S} - \iiint_D f \operatorname{div}(\mathbf{G})\, \mathrm{d}V$.

(c) (5%) Let $f(x,y,z) = 4 - x^2 - y^2 - z^2$ and $\mathbf{G}(x,y,z) = \sin(y+1)\,\mathbf{i} + e^{x+1}\,\mathbf{j} + z^3\,\mathbf{k}$. Use (b) to evaluate

$$\iiint_D \nabla f \cdot \mathbf{G}\, \mathrm{d}V \text{ where } D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 4\}.$$

Solution:

(a) Let $\mathbf{G} = \langle P, Q, R\rangle$. Then $f\mathbf{G} = \langle fP, fQ, fR\rangle$.

$$\begin{aligned}\operatorname{div}(f\mathbf{G}) &= (fP)_x + (fQ)_y + (fR)_z \qquad (1M)\\ &= (f_xP + fP_x) + (f_yQ + fQ_y) + (f_zR + fR_z) \qquad (1M)\\ &= (f_xP + f_yQ + f_zR) + f(P_x + Q_y + R_z) \qquad (1M)\\ &= \nabla f \cdot \mathbf{G} + f\operatorname{div}(\mathbf{G})\end{aligned}$$

(1M) for overall coherence of the proof.

Grading scheme for 4a.

- (1M) Correct definition for divergence.
- (1M) Using product rule for partial derivatives.
- (1M) Grouping the terms appropriately.
- (1M) For overall coherence of the proof.

(b)

$$\begin{aligned}\text{LHS} = \iiint_D \nabla f \cdot \mathbf{G}\, dV &\overset{(a)}{=} \iiint_D \operatorname{div}(f\mathbf{G}) - f\operatorname{div}(\mathbf{G})\, dV \qquad (1M)\\ &= \underline{\iiint_D \operatorname{div}(f\mathbf{G}) dV} - \iiint_D f\operatorname{div}(\mathbf{G})\, dV\\ &\overset{\text{Div. Thm}}{=} \underline{\iint_{\partial D} f\mathbf{G} \cdot d\mathbf{S}} - \iiint_D f\operatorname{div}(\mathbf{G})\, dV = \text{RHS}.\end{aligned}$$

Grading scheme for 4b.

- (1M) Integrate term by term in (a).
- (1M) Indicate clearly which term to apply divergence theorem on and lead to the conclusion.

(c) By using (b), we have

$$\iiint_D \nabla f \cdot \mathbf{G}\, \mathrm{d}V = \iint_{\partial D} f\mathbf{G} \cdot \mathrm{d}\mathbf{S} - \iiint_D f\operatorname{div}(\mathbf{G})\, \mathrm{d}V.$$

(1M) Since ∂D is the sphere $x^2+y^2+z^2 = 4$, we have $\iint_{\partial D} f\mathbf{G} \cdot \mathrm{d}\mathbf{S} = 0$.

(1M) On the other hand, as $f \cdot \operatorname{div}\mathbf{G} = (4 - x^2 - y^2 - z^2)(3z^2)$,

$$\begin{aligned}(3M) \iiint_D f \cdot \operatorname{div}(\mathbf{G})\, \mathrm{d}V &= \iiint_D (4 - x^2 - y^2 - z^2)(3z^2)\, \mathrm{d}V\\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} (4-\rho^2)(3\rho^2\cos^2\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta\\ &= 2\pi \int_0^2 (4\rho^4 - \rho^6)\, d\rho \int_0^{\pi} 3\cos^2\phi\sin\phi\, d\phi\\ &= 2\pi\left(\frac{128}{5} - \frac{128}{7}\right)\cdot 2\end{aligned}$$

Therefore, $\iiint_D \nabla f \cdot \mathbf{G}\, \mathrm{d}V = -4\pi\left(\frac{128}{5} - \frac{128}{7}\right)$.

Grading scheme for 4c.

- (1M) Writing $\iint_{\partial D} f\mathbf{G} \cdot \mathrm{d}\mathbf{S} = 0$
- (1M) Computing $f \cdot \operatorname{div}(\mathbf{G})$ correctly
- (3M) For correct evaluation of $\iiint_D f \cdot \operatorname{div}(\mathbf{G})\,\mathrm{d}V$ (partial credits are available)

5. In this question, you may use, without proof, the fact that $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ is an increasing sequence.

(a) (10%) Determine whether the series

$$\sum_{n=1}^{\infty}(-1)^n\cdot\left(1-n\ln\left(1+\frac{1}{n}\right)\right)$$

is conditionally convergent, absolutely convergent, or divergent.

(b) (5%) Find the interval of convergence of $\sum_{n=1}^{\infty}\frac{n!}{n^n}\cdot x^n$.

Solution:

(a) Let $a_n = 1-n\ln(1+\frac{1}{n}) = \ln e - \ln(1+\frac{1}{n})^n = \ln\left[\frac{e}{(1+\frac{1}{n})^n}\right]$

Since $\{(1+\frac{1}{n})^n\}$ is increasing and converges to e.

We know

(1) $a_n > \ln 1 = 0$ +2分

(2) $a_n > a_{n+1}, \{a_n\}$ is decreasing. +2分

(3) $a_n \to \ln 1 = 0$ as $n\to\infty$ +1分

By Alternating Series Test, $\sum(-1)^n a_n$ converges. (+2分)

Next. $\sum|(-1)^n a_n| = \sum a_n$ by (1)

Observe that when n is large, $n\ln(1+\frac{1}{n}) = n\{\frac{1}{n}-\frac{1}{2}(\frac{1}{n})^2+\frac{1}{3}(\frac{1}{n})^3\cdots\}$

Thus $1-n\ln(1+\frac{1}{n}) = \frac{1}{2}\frac{1}{n}-\frac{1}{3}\frac{1}{n^2}+\cdots$

以下或類似解法正確皆給3分，沒有寫到引用的定理，扣1分

(i) By Maclaurin or Taylor series expansion, $a_n - \frac{1}{2n}-\frac{1}{3n^2}+\cdots$, $\sum a_n$ diverges since $\sum\frac{1}{n^p}$ div.s. if $p\le 1$.

(ii) $\lim_{n\to\infty}\frac{a_n}{\frac{1}{2n}} = 1$, by limit comparison test, $\sum a_n$ diverges since $\sum\frac{1}{n^p}$ div.s. if $p\le 1$.

Ans. $\sum(-1)^n a_n$ conv.s. conditionally.

(b) Let $b_n = \frac{n!}{n^n} > 0$, $\frac{b_{n+1}}{b_n} = (n+1)\frac{n^n}{(n+1)^{n+1}} = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e}$ as $n\to\infty$

By Ratio Test, the radius of convergence is e (+2分)

When $x=\pm e$, $\frac{|b_{n+1}x^{n+1}|}{|b_n x^n|} = \frac{e}{(1+\frac{1}{n})^n} > 1$

That is, $|b_{n+1}x^{n+1}| > |b_n x^n|$. This implies $b_n x^n \nrightarrow 0$ as $n\to\infty$ (+1分)

By nth term test for divergence the series div.s. when $x=e$ (+1分) and $x=-e$ (+1分)

Interval of conv. is $(-e,e)$.

6. Consider the function $f(x) = \int_0^x \frac{1}{\sqrt{1+t^3}}\,dt$.

(a) (3%) Write down the Maclaurin series of $f(x)$ and specify its radius of convergence. (You may express your answer in binomial coefficients $\binom{a}{k}$.)

(b) (3%) What is the value of $f^{(7)}(0)$? Express your answer as a rational number $\frac{a}{b}$ with explicit integers a, b.

(c) (4%) Evaluate $\lim_{x\to 0} \frac{f(x) - x}{(e^{2x^2}-1)\sin(5x^2)}$.

(d) (5%) Express $f(0.5)$ as an alternating series $\sum_{k=0}^{\infty} (-1)^k b_k$ for some $b_k \geq 0$. Prove that $\{b_k\}_{k=0}^{\infty}$ is a decreasing sequence and find $\lim_{k\to\infty} b_k$.

(e) (2%) Hence, determine how many terms of the series in (d) are needed in order to estimate $f(0.5)$ up to an error of 10^{-4}. Justify your estimation.

Solution:

(a)

$$f(x) = \int_0^x (1+t^3)^{-\frac{1}{2}}\,dt = \int_0^x \underbrace{\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \cdot t^{3k}}_{(1M)}\,dt = \underbrace{\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \cdot \frac{x^{3k+1}}{3k+1}}_{(1M)}.$$

(1M) The radius of convergence is 1 (as an integral of a binomial series).

Grading scheme for 6a.

- (1M) Correct use of Binomial series
- (1M) Integrate correctly term-by-term
- (1M) Correct radius of convergence

(b) By Taylor's Theorem, $\underbrace{\frac{f^{(7)}(0)}{7!}}_{(1M)} = \underbrace{\binom{-\frac{1}{2}}{2} \cdot \frac{1}{7}}_{(1M)}$ so $f^{(7)}(0) = 270$ (1M).

Grading scheme for 6b.

- (1M) Correct statement of Taylor's Theorem
- (1M) Correct coefficient of x^7 from (a)
- (1M) Correct answer (as an explicit rational number)

(c) By writing down the leading term of both the numerator and denominator,

$$\lim_{x\to 0} \frac{f(x) - x}{(e^{2x^2}-1)\sin(5x^2)} = \underbrace{\lim_{x\to 0} \frac{-\frac{x^4}{8} + \cdots}{(2x^2+\cdots)(5x^2+\cdots)}}_{(1+1+1M)} = \underbrace{-\frac{1}{80}}_{(1M)}$$

Grading scheme for 6c.

- (1M each ×3) Correct first non-zero term of the Maclaurin series for each factor
- (1M) Correct answer

(d) Note that

$$\begin{aligned}\int_0^{0.5} f(x)\,dx &= \sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\frac{0.5^{3k+1}}{3k+1}\\ &= \sum_{k=0}^{\infty}\frac{-\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-k+1)}{k!}\cdot\frac{0.5^{3k+1}}{3k+1} \qquad \cdots(1M)\\ &= \sum_{k=0}^{\infty}(-1)^k\cdot\underbrace{\frac{\frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+k-1)}{k!}\cdot\frac{0.5^{3k+1}}{3k+1}}_{b_k}\cdots(1M)\end{aligned}$$

- As $\frac{b_{k+1}}{b_k} = \underbrace{\frac{\frac{1}{2}+k}{k+1}\cdot\frac{3k+1}{3k+4}\cdot(0.5)^3}_{(1M)}\underbrace{<1}_{(1M)}$, $\{b_k\}$ is a decreasing seqeunce.
- (1M) As $f(0.5)$ converges, $\lim_{n\to\infty} b_n = 0$ by divergent test.

Grading scheme for 6d.

- (1M) for spelling out the Binomial coefficient
- (1M) for the correct b_k
- (1M) for the ratio b_{k+1}/b_k
- (1M) for mentioning $b_{k+1}/b_k < 1$
- (1M) for writing $\lim_{n\to\infty} b_n = 0$.

(e) By (d), $f(0.5)$ is an alternating series and fulfills the conditions for AST. Let R_k be error incurred by estimating with the k-th partial sum. Then

$$\underbrace{R_k \le |a_{k+1}|}_{1M} = \underbrace{\left|\binom{-\frac{1}{2}}{k+1}\right|\frac{0.5^{3k+4}}{3k+4}}_{1M}$$

For this to be less than 10^{-4}, we can take, for example, $k = 3$.

Grading scheme for 6e.

- (1M) for writing $R_k \le a_{k+1}$ (or b_{k+1})
- (1M) for writing out explicitly the term a_{k+1} or b_{k+1}

Remark : without any valid justification, the choice of k itself doesn't worth any marks.

7. Let $h_n = \sum_{k=1}^{n} \frac{1}{k}$. Consider the sequence $\{t_n\}_{n=1}^{\infty}$ defined by $t_n = h_n - \ln(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$.

(a) (4%) By considering the graph $y = \frac{1}{x}$, interpret t_n as an area and deduce that $\gamma = \lim_{n\to\infty} t_n$ exists.

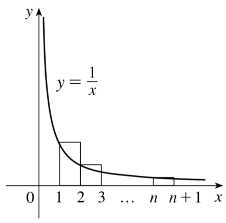
(b) (4%) Let $s_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. By expressing s_{2n} in terms of h_n and canceling out γ, find the value of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$.

(c) (1%) Find $\lim_{n\to\infty} \frac{h_n}{\ln(n)}$.

(d) (5%) Hence, determine whether the series $\sum_{n=1}^{\infty} \frac{h_n}{n^2}$ converges or not.

Solution:

(a) $\boxed{2\% : \text{Prove that } t_n \geq 0.}$ Consider the *upper Riemann sum* :

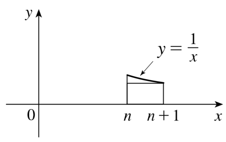


The sum of the areas of the rectangles is $h_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. This is larger than the area under curve $\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$. Consequently, $t_n = h_n - \ln(n) \geq \ln(n+1) - \ln(n) \geq 0$.

$\boxed{2\% : \text{Prove that } \{t_n\} \text{ is decreasing.}}$ Note that

$$t_{n+1} - t_n = \frac{1}{n+1} - \ln(n+1) + \ln(n) = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x}\,dx$$

which is ≤ 0 because it represents the difference of the area of a 'lower' rectangle and the area under curve on $[n, n+1]$:



Hence, by monotone convergence theorem, $\lim_{n\to\infty} t_n$ exists.

(b) Since $s_{2n} = 1 - \frac{1}{2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \left(1 + \frac{1}{2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) = h_{2n} - h_n$ (2%),
we have $s_{2n} = t_{2n} + \ln(2n) - t_n - \ln(n) = t_{2n} - t_n + \ln 2$ and hence

$$\lim_{n\to\infty} s_{2n} = \gamma - \gamma + \ln 2 = \ln 2. (2\%)$$

Consequently, as the alternative harmonic series converges, we have $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{2n} = \ln 2$.

(c) $\lim_{n\to\infty} \frac{h_n}{\ln n} = \lim_{n\to\infty} \left(\frac{t_n}{\ln n} + 1\right) = 0 + 1 = 1$ (1%).

(d) (1%) Let $a_n = \frac{h_n}{n^2} \geq 0$ and $b_n = \frac{\ln n}{n^2} \geq 0$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{h_n}{\ln n} = 1 > 0$. Therefore, by Limit Comparison Test, it suffices to determine the convergence of $\sum b_n$.
(4%) Let $f(x) = \frac{\ln x}{x^2}$. Then on $[2, \infty)$, $f(x)$ is positive, continuous and decreasing and moreover,

$$\int_2^t \frac{\ln x}{x^2}\,dx = \frac{1 + \ln 2}{2} - \frac{\ln t + 1}{t}$$

which converges to $\dfrac{1+\ln 2}{2}$ as $t \to \infty$. By Integral Test, $\sum b_n$ converges and hence by Limit Comparison Test, $\sum a_n$ converges as well.

Alternatively, for n large enough, we have $\ln n \leq \sqrt{n}$ and hence, $b_n \leq \dfrac{1}{n^{1.5}}$. Since $\sum \dfrac{1}{n^{1.5}}$ converges, the series $\sum b_n$ converges by Direct Comparison Test and hence by Limit Comparison Test, $\sum a_n$ converges as well.