1112 模組01-05班 微積分3 期考解答和評分標準

1. Consider the function $f(x,y) = \begin{cases} \dfrac{x^5 y}{x^6+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

(a) (5%) Let $\mathbf{u} = \langle a,b \rangle$ be a unit vector. Use the definition of directional derivative to find $D_{\mathbf{u}} f(0,0)$.

(b) (2%) Write down the linearization $L(x,y)$ of $f(x,y)$ at $(0,0)$.

(c) (6%) Prove that $f(x,y)$ is not differentiable at $(0,0)$ by showing that the limit

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} \text{ does not exist.}$$

(d) (5%) Let $\mathbf{r}(t) = \langle t, t^{\frac{4}{3}} \rangle$ be a curve on $\mathbb{R}^2$. Find, by using the definition of derivative,

$$\frac{d}{dt} f(\mathbf{r}(t))\Big|_{t=0}.$$

Is it true that $\dfrac{d}{dt} f(\mathbf{r}(t))\Big|_{t=0} = \nabla f(0,0)\cdot \mathbf{r}'(0)$ in this case?

Solution:

(a)

$$D_{\mathbf{u}} f(0,0) = \lim_{t\to0} \frac{f(at,bt)-f(0,0)}{t} \qquad (+2)$$

$$= \lim_{t\to0} \frac{a^5 b\, t^6}{a^6 t^7 + b^4 t^5} = \lim_{t\to} \frac{a^5 b\, t}{a^6 t^2 + b^4} \qquad (+2)$$

If $b \neq 0$, $D_{\mathbf{u}} f(0,0) = \lim_{t\to0} \dfrac{a^5 b\, t}{a^6 t^2 + b^4} = \dfrac{0}{b^4} = 0$.

If $b = 0$, then $f(at,bt) = 0$ for all t and $D_{\mathbf{u}} f(0,0) = \lim_{t\to0} \dfrac{f(at,bt)-f(0,0)}{t} = \lim_{t\to0} \dfrac{0}{t} = 0$.

In conclusion, $D_{\mathbf{u}} f(0,0) = 0$ for any $\mathbf{u}$. (+1)

(b) From part (a), we have $f_x(0,0) = D_{\mathbf{i}} f(0,0) = 0$ and $f_y(0,0) = D_{\mathbf{j}} f(0,0) = 0$. (+1)

Hence the linearization of f at $(0,0)$ is

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0. \qquad (+1)$$

(c) Let

$$g(x,y) = \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} = \frac{x^5 y}{(x^6+y^4)\sqrt{x^2+y^2}}. \qquad \left(+1 \text{ for simplifing } \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}}\right)$$

$g(t^2,t^3) = \dfrac{t^{13}}{(t^{12}+t^{12})\sqrt{t^4+t^6}} = \dfrac{t^{13}}{2t^{14}\sqrt{1+t^2}} = \dfrac{1}{2t\sqrt{1+t^2}}$ approaches infinity as $t \to 0^+$ which means that g approaches infinity as (x,y) approaches $(0,0)$ along the curve $x^3 = y^2$. Hence the limit $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist. And $f(x,y)$ is NOT differentiable at $(0,0)$. (+5 for finding a path along which the limit of g doesn't exist.)

Students may approach $(0,0)$ along other paths. For example,

$g(x,x^2) = \dfrac{x^7}{(x^6+x^8)\sqrt{x^2+x^4}} \to 1$ as $x \to 0^+$ and $g(x,x^2) \to -1$ as $x \to 0^-$.

This also shows that $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist.

(d)

$$f(\mathbf{r}(t)) = f(t,t^{\frac{4}{3}}) = \frac{t^5 t^{\frac{4}{3}}}{t^6 + t^{\frac{16}{3}}} = \frac{t}{t^{\frac{2}{3}}+1} \qquad \text{for } t \neq 0, \quad (+2)$$

and $f(\mathbf{r}(0)) = f(0,0) = 0$. Hence, by definition

$$\frac{d}{dt} f(\mathbf{r}(t))\big|_{t=0} = \lim_{t\to0} \frac{f(\mathbf{r}(t))-f(\mathbf{r}(0))}{t} = \lim_{t\to0} \frac{1}{t^{\frac{2}{3}}+1} = 1 \qquad (+2)$$

However, $\nabla f(0,0) = 0\,\mathbf{i} + 0\,\mathbf{j}$ and $\mathbf{r}'(0) = \mathbf{i}$. Thus

$$\nabla f(0,0)\cdot\mathbf{r}'(0) = 0 \neq \frac{d}{dt} f(\mathbf{r}(t))\big|_{t=0}. \qquad (+1)$$

2. Let $F(x,y,z) = x^2 + y^2 + z^2$ and $G(x,y,z) = z^3 - 3xy + y^2$. Let C be the curve of intersection of the level surfaces $F(x,y,z) = 9$ and $G(x,y,z) = 6$.

(a) (6%) Find a parametization of the tangent line of C at $(1,2,2)$.

(b) Near $(1,2,2)$, the curve defines $y = y(x)$ and $z = z(x)$ as differentiable functions in x.

(i) (4%) Find $\left.\frac{d}{dx}F(x,y(x),z(x))\right|_{x=1}$ and $\left.\frac{d}{dx}G(x,y(x),z(x))\right|_{x=1}$. Express your answers in $y'(1)$ and $z'(1)$.

(ii) (3%) Hence, find the values of $y'(1)$ and $z'(1)$.

(c) (5%) It is known that a differentiable function $H(x,y,z)$, when restricted to the surface $F(x,y,z) = 9$, attains its absolute maximum value at $(1,2,2)$ and $H_y(1,2,2) = -2$. Use linearization to estimate the value of $H(1.1,1.9,2.1) - H(1,2,2)$.

Solution:

(a) Let the tangent line of C at $(1,2,2)$ be L. Since curve C lies on the level surface $F(x,y,z) = 9$, L lies on the tangent plane of $F(x,y,z) = 9$ at $(1,2,2)$. Moreover, $\nabla F(1,2,2)$ is a normal vector of the tangent plane. Thus we conclude that $\nabla F(1,2,2)$ and L are orthogonal. Similarly, L lies on the tangent plane of $G(x,y,z) = 6$ and $\nabla G(1,2,2)$ and L are orthogonal. Therefore, L is parallel to

$$\nabla F(1,2,2) \times \nabla G(1,2,2). \qquad (+2)$$

$$\nabla F(1,2,2) = (2x,2y,2z)|_{(1,2,2)} = (2,4,4) \qquad (+1)$$

$$\nabla G(1,2,2) = (-3y,-3x+2y,3z^2)\big|_{(1,2,2)} = (-6,1,12) \qquad (+1)$$

Hence L is parallel to

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 4 \\ -6 & 1 & 12 \end{vmatrix} = 44\mathbf{i} - 48\mathbf{j} + 26\mathbf{k}. \qquad (+1)$$

Hence a parametrization of the tangent line L is

$$x(t) = 1 + 44t, \quad y(t) = 2 - 48t, \quad z(t) = 2 + 26t. \quad t \in \mathbf{R} \qquad (+1)$$

(b) (i) Near $(1,2,2)$, the curve C can be described as $(x,y(x),z(x))$. Hence $y(1) = 2$, $z(1) = 1$.

$$\left.\frac{d}{dx}F(x,y(x),z(x))\right|_{x=1} = F_x(1,2,2)\cdot 1 + F_y(1,2,2)\cdot y'(1) + F_z(1,2,2)\cdot z'(1) \quad (+1)$$

$$= 2 + 4y'(1) + 4z'(1) \qquad (+1)$$

$$\left.\frac{d}{dx}G(x,y(x),z(x))\right|_{x=1} = G_x(1,2,2)\cdot 1 + G_y(1,2,2)\cdot y'(1) + G_z(1,2,2)\cdot z'(1) \quad (+1)$$

$$= -6 + y'(1) + 12z'(1) \qquad (+1)$$

(ii) Since the curve $C : (x,y(x),z(x))$ lies on the level surfaces $F(x,y,z) = 9$ and $G(x,y,z) = 6$, we have $F(x,y(x),z(x)) = 9$ and $G(x,y(x),z(x)) = 6$. Hence

$$\begin{cases} \left.\frac{d}{dx}F(x,y(x),z(x))\right|_{x=1} = 0 = 2 + 4y'(1) + 4z'(1) \\ \left.\frac{d}{dx}G(x,y(x),z(x))\right|_{x=1} = 0 = -6 + y'(1) + 12z'(1) \end{cases} \qquad (+1)$$

Solve the above system of equations, we get

$$y'(1) = -\frac{12}{11} \quad (+1), \qquad z'(1) = \frac{13}{22} \quad (+1)$$

(c) Since under the constraint $F(x,y,z) = 9$, $H(x,y,z)$ obtains absolute maximum at $(1,2,2)$ and $\nabla F(1,2,2) \neq \mathbf{0}$, from the method of Lagrange multipliers we know that

$$\nabla H(1,2,2) = \lambda \nabla F(1,2,2) = \lambda(2,4,4) \qquad (+2)$$

Given that $H_y(1,2,2) = -2$, we conclude that $\lambda = -\frac{1}{2}$ and $\nabla H(1,2,2) = (-1,-2,-2)$. $\quad (+1)$

By the linearization of H at $(1,2,2)$, we can estimate

$$H(1.1,1.9,2.1) - H(1,2,2) \approx H_x(1,2,2)\cdot(1.1-1) + H_y(1,2,2)\cdot(1.9-2) + H_z(1,2,2)\cdot(2.1-2) \qquad (+1)$$

$$= (-1)\cdot(0.1) + (-2)\cdot(-0.1) + (-2)\cdot(0.1) = -0.1. \qquad (+1)$$

3. (14%) It is known that the plane $x + y - 2z = 5$ and the cylinder $3x^2 + 2xy + 3y^2 = 16$ intersect at an ellipse Γ centered at $\left(0, 0, -\frac{5}{2}\right)$. Apply the method of Lagrange multipliers to find the maximum and minimum distances of Γ from its center.

Solution:

對以下 2 種可能解答方式，請依相同原則給分。

(1) Find the max and min values of

$$f(x,y,z) = x^2 + y^2 + \underline{\left(z - \left(-\frac{5}{2}\right)\right)^2} \quad \text{subject to}$$
$$g(x,y,z) = x + y - 2z\underline{-5} = 0 \quad \text{and}$$
$$h(x,y,z) = 3x^2 + 2xy + 3y^2 - 16 = 0$$

(2) Find the max and min values of

$$\tilde{f}(x,y,z) = x^2 + y^2 + \underline{z^2} \text{ subject to } \begin{cases} \tilde{g}(x,y,z) = x + y - 2\underline{z = 0} \\ h(x,y,z) = 3x^2 + 2xy + 3y^2 - 16 = 0 \end{cases}$$

(只列出 case (1))

By the method of Langrange multipliers, we need to solve (x, y, z, μ, λ) satisfying

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h & \to 3分 \ (1) \\ g = 0 & \to 1分 \ (2) \\ h = 0 & \to 1分 \ (3) \end{cases}$$

其中 (1) **為**

$$\begin{cases} 2x = \lambda \cdot 1 + \mu \cdot (6x + 2y) & \to 1分 \ (4) \\ 2y = \lambda \cdot 1 + \mu \cdot (2x + 6y) & \to 1分 \ (5) \\ 2(z + \frac{5}{2}) = \lambda \cdot (-2) + \mu \cdot 0 & \to 1分 \ (6) \end{cases}$$

以上全寫完整且正確，才繼續批改
大致查看，確認有意義的解 (1), (2), (3)**，就跳至最後檢查答案，給分**

By (3) or (6)，$\lambda = -(z + \frac{5}{2})$ $\therefore \begin{cases} 2x + z + \frac{5}{2} = 2\mu(3x + y) & (7) \\ 2y + z + \frac{5}{2} = 2\mu(x + 3y) & (8) \end{cases}$

Since $(4)\cdot x + (5)\cdot y + (6)\cdot(z + \frac{5}{2}) = 2\left[x^2 + y^2 + (z + \frac{5}{2})^2\right] = 2\cdot 16\cdot\mu$, μ can not be zero. $\frac{(7)}{(8)}$ gives $(x-y)(x+y-(z+\frac{5}{2})) = 0$

If $x = y$, $\underline{(x,y,z) = (\pm\sqrt{2}, \pm\sqrt{2}, \pm\sqrt{2} - \frac{5}{2})}$ 2 分 $\quad$ min distance $= \sqrt{6}$ 1 分

If $x + y - (z + \frac{5}{2}) = 0$, $\underline{(x,y,z) = (\pm 2, \mp 2, -\frac{5}{2})}$ 2 分 $\quad$ max distance $= \sqrt{8} = 2\sqrt{2}$ 1 分

4. (a) (8%) Evaluate $\int_{-2}^{0}\int_{-\frac{y}{2}}^{1} e^{-4(x^3+x^2)}\,\mathrm{d}x\mathrm{d}y + \int_{0}^{3}\int_{\sqrt{\frac{y}{3}}}^{1} e^{-4(x^3+x^2)}\,\mathrm{d}x\mathrm{d}y.$

(b) (8%) Let D be the region in the first quadrant that is bounded by $x^2+3y^2=1$, $x^2+3y^2=5$, $y=x$ and the x-axis. Evaluate $\iint_D \cos(x^2+3y^2)\,\mathrm{d}A.$

Solution:

(a) Notice that the integrants of the two integrals are the same and the union D of the domains of the integrals is given by

$$\begin{aligned} D &= \{-2 \le y \le 0, -y/2 \le x \le 1\} \cup \{0 \le y \le 3, \sqrt{y/3} \le x \le 1\} \\ &= \{(x,y) \,|\, 0 \le x \le 1, -2x \le y \le 3x^2\}, \end{aligned}$$

which is a type I region. We have, by Fubini's theorem,

$$\begin{aligned} \iint_D e^{-4(x^3+x^2)}\mathrm{d}A &= \int_0^1 \int_{-2x}^{3x^2} e^{-4(x^3+x^2)}\mathrm{d}y\mathrm{d}x \quad (+5) \\ &= \int_0^1 (3x^2+2x)e^{-4(x^3+x^2)}\mathrm{d}x \quad (+1) \\ &= \frac{-1}{4}e^{-4(x^3+x^2)}\Big|_0^1 \quad (+2) \\ &= \frac{1}{4}(1-e^{-8}). \end{aligned}$$

[Here, correctly identify the type I region D: (+3), and set up the integral: (+2).]

(b) Let $u=x, v=\sqrt{3}y$ and then $u=r\cos\theta, v=r\sin\theta$, which together give

$$x = r\cos\theta, y = \frac{1}{\sqrt{3}}r\sin\theta. \quad (+3)$$

The conditions $x^2+3y^2=1, x^2+3y^2=5, y=x$, and $y=0$ (the x-axis) become $r=1, r=\sqrt{5}, \theta=\pi/3$, and $\theta=0$, respectively (+2); the Jacobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \frac{1}{\sqrt{3}}\sin\theta & \frac{1}{\sqrt{3}}r\cos\theta \end{vmatrix} = \frac{1}{\sqrt{3}}r \quad (+2).$$

$$\begin{aligned} \iint_D \cos(x^2+3y^2)\,dA &= \frac{1}{\sqrt{3}}\int_0^{\pi/3}\int_1^{\sqrt{5}} r\cos r^2\,dr d\theta \\ &= \frac{\pi}{3\sqrt{3}}\Big[\frac{1}{2}\sin r^2\Big]_1^{\sqrt{5}} \quad (+1) \\ &= \frac{\pi}{6\sqrt{3}}(\sin 5 - \sin 1). \end{aligned}$$

[Find the correct change of variables: (+3). The linear transformation $u=x, v=\sqrt{3}y$: (+2), and the polar coordinates: (+1).

Find the correct region after the change of coordinates: (+2). The conditions $u^2+v^2=1,5, v=\sqrt{3}u, v=0$ for u,v: (+1).

Find the correct Jacobian: (+2). For the linear map: (+1), and the polar coordinates: (+1).

Find the correct antiderivatives: (+1).]

5. (a) Let U be the solid region enclosed by the surfaces $x^2+y^2+z^2=6$ and $z=x^2+y^2$.

(i) (5%) Express the volume of U as an iterated integral in cylindrical coordinates :

$$\text{Volume}(U)=\int_a^b\int_{c(\theta)}^{d(\theta)}\int_{e(r,\theta)}^{f(r,\theta)} g(r,\theta,z)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

(ii) (4%) Hence, find the volume of U.

(b) Let S be the solid that lies below the sphere $x^2+y^2+z^2=1$ and above the cone $z=\sqrt{3x^2+3y^2}$ with density function $\rho(x,y,z)=\sqrt{z}$.

(i) (5%) Express the mass of S as an iterated integral in spherical coordinates :

$$\text{Mass}(S)=\int_a^b\int_c^d\int_e^f g(\rho,\theta,\phi)\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

(ii) (4%) Hence, find the mass of S.

Solution:

(a) (i) $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} r\,dz\,dr\,d\theta$

Grading scheme for Q5a i

- (4M) For the integration limits [Partial credits available]
- (1M) Jacobian of cylindrical coordinates

Remarks.

(i) -1M overall for students swapping $\sqrt{6-r^2}$ with r^2.

(ii) -1M for each mistake incurred in writing down the integration limits.

(ii)

$$\begin{aligned}\text{Volume}(U)&=2\pi\int_0^{\sqrt{2}} r\sqrt{6-r^2}-r^3\,dr\\&=2\pi\left[-\frac{1}{3}(6-r^2)^{\frac{3}{2}}-\frac{r^4}{4}\right]_{r=0}^{r=\sqrt{2}} &\cdots(2\text{M})\\&=2\pi\left(-\frac{8}{3}-1+\frac{6^{\frac{3}{2}}}{3}\right)=2\pi\left(\frac{6^{\frac{3}{2}}-11}{3}\right) &\cdots(2\text{M}).\end{aligned}$$

Grading scheme for Q5b ii

- (2M) For an antiderivative of $r\sqrt{6-r^2}-r^3$
- (2M) Correct answer

Remarks.

(i) No marks in this part for students who forget the Jacobian in (a)(i).

(b) (i) $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^1\sqrt{\rho\cos\phi}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Grading scheme for Q5b i

- (3M) For the integration limits [Partial credits available]
- (1M) For the integrand
- (1M) Jacobian of spherical coordinates

(ii) The mass can be computed as

$$\left(\int_0^{2\pi} 1\,d\theta\right)\cdot\left(\int_0^1 \rho^{\frac{5}{2}}\,d\rho\right)\cdot\left(\int_0^{\frac{\pi}{6}} \sin\phi\sqrt{\cos\phi}\,d\phi\right) = 2\pi\cdot\frac{2}{7}\cdot\left[-\frac{2}{3}(\cos\phi)^{\frac{3}{2}}\right]_0^{\frac{\pi}{6}}$$

$$= 2\pi\cdot\frac{2}{7}\cdot\left(\frac{2}{3}-\frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^{\frac{3}{2}}\right)$$

Grading scheme for Q5b ii

- (2M) Anti-derivative of $\sin\phi\sqrt{\cos\phi}$ [Partial credits available]
- (2M) Correct answer [Partial credits available]

6. (a) (8%) Let $f(x)$ be a continuous function on $\mathbb{R}$ and T be the triangular region on the xy-plane whose vertices are $(0,0)$, $(2,0)$ and $(0,3)$. Using a suitable change of variables, show that

$$\iint_T f(3x+2y)\,\mathrm{d}A = \frac{1}{6}\int_0^6 uf(u)\,\mathrm{d}u.$$

(b) Let U be the solid that is below the plane $z = 3x+2y$ and above the region T in (a) on the xy-plane. Consider

$$I = \iiint_U \cos\left(\frac{z^3}{3} - 36z\right)\mathrm{d}V.$$

(i) (3%) Find a function f such that

$$I = \iint_T f(3x+2y)\,\mathrm{d}A.$$

You may express f as an integral (**Do <u>not</u> evaluate it !**).

(ii) (5%) Hence, use (a) and then Fubini's Theorem to evaluate I.

Solution:

(a) This is a proof-based question so we expect the candidates to demonstrate every step clearly. There are many possible change of variables. One of which is the following :

(1M) Let $u = 3x+2y$ and $v = y$.

(1M) Then $x = \frac{1}{3}u - \frac{2}{3}v$ and $y = v$.

(2M) Jacobian is $\begin{vmatrix} \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 \end{vmatrix} = \frac{1}{3}$

(2M)The given region becomes the region enclosed by $u = 2v$, $v = 0$ and $u = 6$.

As a result, we have

$$\iint_T f(3x+2y)\,dA = \int_0^6 \int_0^{\frac{1}{2}u} f(u)\cdot\frac{1}{3}\,dv\,du = \frac{1}{6}\int_0^6 uf(u)\,du.$$

(2M for the overall coherence and completeness of the argument)

<u>Grading scheme for Q6a</u>

- (1M) For making a reasonable substitution $u = u(x,y)$ and $v = v(x,y)$
- (1M) For solving $x = x(u,v)$ and $y = y(u,v)$
- (2M) For correct Jacobian
- (2M) For transforming the region correctly
- (2M) For the overall coherence and completeness of the argument

(b) (i) $I = \iint_T \int_0^{3x+2y} \cos\left(\frac{z^3}{3} - 36z\right) dz\,dA$ so $f(u) = \int_0^u \cos\left(\frac{z^3}{3} - 36z\right) dz$.

<u>Grading scheme for Q6bi</u>

- All or nothing. Except for obvious typos.

(ii)

$$\begin{aligned}
\iiint_U \cos(z^3 - 12z)\,\mathrm{d}V &= \iint_T \left(\int_0^{3x+2y} \cos\left(\frac{z^3}{3} - 36z\right) dz\right) dA \\
&\overset{(a)}{=} \frac{1}{6}\int_0^6 \int_0^u u\cos\left(\frac{z^3}{3} - 36z\right) dzdu \qquad \cdots\text{(1M)} \\
&\overset{\text{Fubini}}{=} \frac{1}{6}\int_0^6 \int_z^6 u\cos\left(\frac{z^3}{3} - 36z\right) dudz \qquad \cdots\text{(2M)} \\
&= \frac{1}{12}\int_0^6 (36 - z^2)\cos\left(\frac{z^3}{3} - 36z\right) dz \\
&= \frac{1}{12}\left[\sin\left(36z - \frac{z^3}{3}\right)\right]_0^6 \\
&= \frac{\sin(144)}{12} \qquad \cdots\text{(2M)}
\end{aligned}$$

Grading scheme for Q6bii

- (1M) for using (a) correctly
- (2M) for applying Fubini correctly
- (2M) for the correct answer