1. Let $G(x) = \int_1^{e^{2x}} f(t)\,dt$ where $f(x)$ is a continuous function.
   (a) (5%) Compute $G'(x)$.
   (b) (5%) Suppose that $G(x) = \ln(1+x^4)$. Find $f(e^2)$.

Solution:

(a) $G'(x) = f(e^{2x}) \cdot 2 \cdot e^{2x}$

Wrong answers with corresponding credits

$G'(x) = f(e^{2x}) \cdot e^{2x}$ 3 pts

$G'(x) = f(e^{2x})e^{2x} \cdot 2 - f(1)$ 3 pts

$G'(x) = f(e^{2x})$ 1 pt

(b) $G'(x) = f(e^{2x}) \cdot 2 \cdot e^{2x} = \dfrac{4x^3}{1+x^4}$

2 pts for differentiating $\ln(1+x^4)$.

Let $x = 1$, $G'(1) = f(e^2) \cdot 2 \times e^2 = \dfrac{4}{2}$

1 pt for plugging in $x = 1$.

Hence $f(e^2) = \dfrac{1}{e^2}$.

2 pts for the final answer.

2. (a) (10%) Compute $\int_0^1 x^2 \tan^{-1} x \, dx$.

(b) (10%) Compute $\int_1^{\frac{3}{2}} \sqrt{1-(x-1)^2} \, dx$.

Solution:

(a)

$$\int_0^1 x^2 \tan^{-1} x \, dx = \frac{1}{3}\int_0^1 \tan^{-1} x \, d(x^3) = \left[\frac{x^3}{3}\tan^{-1} x\right]_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{1+x^2}$$

$$= \frac{\pi}{12} - \frac{1}{3}\int_0^1 \left(x - \frac{x}{1+x^2}\right) dx = \frac{\pi}{12} - \frac{1}{3}\left[\frac{x^2}{2} - \frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \frac{\pi}{12} - \frac{1}{6} + \frac{\ln 2}{6}$$

□

(b)

$$\int_1^{3/2} \sqrt{1-(x-1)^2} \, dx = \int_1^{3/2} \sqrt{1-(x-1)^2} \, d(x-1) = \int_0^{1/2} \sqrt{1-u^2} \, du$$

Set $u = \sin\theta$, $du = \cos\theta \, d\theta$, $-\pi/2 \le \theta \le \pi/2$

$$= \int_0^{\pi/6} \cos^2\theta \, d\theta = \frac{1}{2}\int_0^{\pi/6} 1 + \cos(2\theta) \, d\theta = \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

□

Grading:

There are many ways for the student to get the correct answer. Read their work, -2% for each minor mistake. -3% for any antiderivative mistake.

If the student did not arrive at the correct answer (unfinished or major mistake), they get +3% for the first correct integration technique and +1% for the first correct antiderivative.

If the student mis-copied the problem, determine if the new integral is of similar difficulty. Grade normally if it is, otherwise max 5%.

3. Let $f(x) = \dfrac{-8x^2 - 7x + 3}{(x+1)(x+2)(x^2+1)}$.

(a) (6%) Write $f(x)$ as $\dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{Cx+D}{x^2+1}$. Find constants A, B, C, D.

(b) (8%) Compute $\int f(x)\, dx$.

(c) (6%) Compute $\int_0^\infty f(x)\, dx$.

Solution:

(a) (M1) By

$$\frac{-8x^2 - 7x + 3}{(x+1)(x+2)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1},$$

we have

$$-8x^2 - 7x + 3 = A(x+2)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x+2) \ (1\%).$$

When $x = -1$, we have

$$-8(-1)^2 - 7(-1) + 3 = A(-1+2)((-1)^2+1) \ \Rightarrow\ A = 1 \ (1\%).$$

When $x = -2$, we have

$$-8(-2)^2 - 7(-2) + 3 = B(-2+1)((-2)^2+1) \ \Rightarrow\ B = 3 \ (1\%).$$

Then

$$\begin{aligned} -8x^2 - 7x + 3 &= (x+2)(x^2+1) + 3(x+1)(x^2+1) + (Cx+D)(x+1)(x+2) \\ &= (4+C)x^3 + (5+3C+D)x^2 + (4+2C+3D)x + (5+2D) \ (1\%). \end{aligned}$$

So we obtain $C = -4$ (1%) and $D = -1$ (1%).

(M2) By

$$\frac{-8x^2 - 7x + 3}{(x+1)(x+2)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1},$$

we have

$$\begin{aligned} & -8x^2 - 7x + 3 = A(x+2)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x+2) \\ = \ & (A+B+C)x^3 + (2A+B+3C+D)x^2 + (A+B+2C+3D)x + (2A+B+2D) \ (1\%). \end{aligned}$$

So

$$A+B+C = 0,\ 2A+B+3C+D = -8,\ A+B+2C+3D = -7,\ 2A+B+2D = 3 \ (1\%).$$

Then we obtain that $A = 1$ (1%), $B = 3$ (1%), $C = -4$ (1%) and $D = -1$ (1%).

(b)

$$\begin{aligned} \int f(x)\, dx &= \int \frac{1}{x+1} + \frac{3}{x+2} - \frac{4x}{x^2+1} - \frac{1}{x^2+1}\, dx \\ &= \ln|x+1| \ (1\%) + 3\ln|x+2| \ (1\%) - 2\ln|x^2+1| \ (3\%) - \tan^{-1} x \ (2\%) + C \ (1\%) \end{aligned}$$

(c)

$$\begin{aligned} \int_0^\infty f(x)\, dx &= \lim_{b\to\infty} \int_0^b f(x)\, dx \ (2\%) \\ &= \lim_{b\to\infty} \ln\left|\frac{(x+1)(x+2)^3}{(x^2+1)^2}\right| - \tan^{-1} x \Bigg]_0^b \\ &= \lim_{b\to\infty} \ln\left|\frac{(b+1)(b+2)^3}{(b^2+1)^2}\right| - \tan^{-1} b - \ln 8 \ (1\%) \end{aligned}$$

Since

$$\lim_{b\to\infty} \frac{(b+1)(b+2)^3}{(b^2+1)^2} = 1 \ (2\%)$$

we have

$$\int_0^\infty f(x)\,dx$$
$$= \lim_{b\to\infty} \ln\left|\frac{(b+1)(b+2)^3}{(b^2+1)^2}\right| - \tan^{-1} b - \ln 8 = -\frac{\pi}{2} - \ln 8 \ (1\%)$$

4. Let X be the random variable representing the life-time(years) of a type of light bulb. Suppose that the probability density function of X is $f(x) = \begin{cases} \frac{1}{5}e^{-x/5} & ,\text{if } x \geq 0 \\ 0 & ,\text{if } x < 0 \end{cases}$.

(a) (7%) Compute the expected value, $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$.

(b) (4%) Find the probability, $\mathbf{P}(2X + 1 \leq 13)$.

(c) (5%) Let $Y = 2X + 1$. Write down the distribution function of Y, $F(y) = \mathbf{P}(Y \leq y)$, as an integral. Find the probability density function of Y, $\frac{d}{dy}F(y)$.

Solution:

(a)

$$\begin{aligned} E(x) &= \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \frac{1}{5} e^{-\frac{x}{5}} dx = \lim_{t\to\infty} \left(\int_0^t \frac{x}{5} e^{-\frac{x}{5}} dx \right) \\ &= \lim_{t\to\infty} \left(-xe^{-\frac{x}{5}} \Big|_0^t + \int_0^t e^{-\frac{x}{5}} dx \right) \qquad \text{3 pts for integration by parts} \\ &= \lim_{t\to\infty} \left(-te^{-\frac{t}{5}} - 5e^{-\frac{x}{5}} \Big|_0^t \right) \qquad \text{2 pts for integrating } e^{-\frac{x}{5}} \\ &= \lim_{t\to\infty} \left(\frac{-t}{e^{t/5}} - 5e^{-\frac{t}{5}} + 5 \right) \end{aligned}$$

$\lim_{t\to\infty} \frac{t}{e^{\frac{t}{5}}} \overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=\!=}} \lim_{t\to\infty} \frac{1}{\frac{1}{5}e^{t/5}} = 0$. $\lim_{t\to\infty} e^{-t/5} = 0$

Hence $E(x) = \lim_{t\to\infty} \left(\frac{-t}{e^{t/5}} - 5e^{-t/5} + 5 \right) = 5$ 2 pts for computing limits

(b)

$$\begin{aligned} \mathbf{P}(2X + 1 \leq 13) &= \mathbf{P}(X \leq 6) \qquad \text{1 pt} \\ &= \int_0^6 \frac{1}{5} e^{-\frac{x}{5}} dx = -e^{-\frac{x}{5}} \Big|_0^6 \qquad \text{2 pts for } \int \frac{1}{5} e^{-\frac{x}{5}} dx = -e^{-\frac{x}{5}} + C \\ &= -e^{-6/5} + 1 \qquad \text{1 pt for the final answer} \end{aligned}$$

(c) $F(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(2X + 1 \leq y) = \mathbf{P}(X \leq \frac{y-1}{2}) = \begin{cases} \int_0^{\frac{y-1}{2}} \frac{1}{5} e^{-\frac{x}{5}} dx & ,\text{if } y \geq 1 \\ 0 & ,\text{if } y < 1 \end{cases}$ 2 pts

Then the probability density function of Y is $\frac{d}{dy}F(y) = \begin{cases} \frac{1}{10} e^{-\frac{y-1}{10}} & ,\text{if } y \geq 1 \\ 0 & ,\text{if } y < 1 \end{cases}$

3 pts for applying F.T.C.

If Students do not discuss the case $y < 1$, they have 1 pt off.

5. (a) (10%) Assume that the rate of change of the unit price of a commodity is proportional to the difference between the demand and supply, so that $\frac{dp}{dt} = k(D(p) - S(p))$, where $k > 0$ is a constant. Suppose that $D(p) = 60 - 3p$, $S(p) = 10 + 2p$ and $p(0) = 5$. Solve $p(t)$.

(b) (10%) Solve the differential equation $\frac{dy}{dx} + \frac{y}{x \ln x} = \frac{x}{\ln x}$ for $x \geq 3$ with $y(3) = 0$.

Solution:

(a) From $\frac{dp}{dt} = k(D(p) - S(p))$, we have $\frac{dp}{dt} = k(50 - 5p)$. Thus

$$\begin{aligned} \frac{p'}{50-5p} &= k. \\ &\Rightarrow \int \frac{p'}{50-5p} dt = \int k dt \\ &\Rightarrow \frac{-1}{5} \ln|50-5p| = kt + C \\ &\Rightarrow \ln|50-5p| = -5kt + C \\ &\Rightarrow 50 - 5p = Ae^{-5kt} \text{where } A = \pm e^{-5C} \end{aligned}$$

Since $p(0) = 5$, $25 = A$. Hence $p(t) = 10 - 5e^{-5kt}$.
(1 point for $\frac{dp}{dt} = k(50 - 5p)$,
2 points for $\frac{p'}{50-5p} = k$,
2 points for $\ln|50 - 5p| = -5kt + C$
2 points for $50 - 5p = Ae^{-5kt}$
2 points for $A = 25$.
1 point for $p(t) = 10 - 5e^{-5kt}$.

(b) The integrator $I(x)$ is $e^{\int \frac{1}{x \ln x} dx}$. We compute

$$\int \frac{1}{x \ln x} dx = \ln(\ln x) + C.$$

Thus $I(x) = e^{\ln(\ln x)} = \ln x$. We have that $(\ln x \cdot y)' = x \Rightarrow \ln x \cdot y = \frac{1}{2}x^2 + C \Rightarrow y = \frac{x^2}{2\ln x} + \frac{C}{\ln x}$. Since $y(3) = 0$, $\frac{9}{2\ln 3} + \frac{C}{\ln 3} = 0 \Rightarrow C = \frac{-9}{2}$. Hence $y = \frac{x^2}{2\ln x} + \frac{-9}{2\ln x}$.
(1 point for $I(x) = e^{\int \frac{1}{x \ln x} dx}$.
2 points for $I(x) = \ln x$.
2 points for $(\ln x \cdot y)' = x$.
2 points for $\ln x \cdot y = \frac{1}{2}x^2 + C$.
2 points for $C = \frac{-9}{2}$.
1 point for $y = \frac{x^2}{2\ln x} + \frac{-9}{2\ln x}$.

6. (a) (6%) Write down the Taylor series of $\int_0^x \sin(t^2)\,dt$ at $x = 0$, given that $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$.

(b) (4%) Write down the Taylor series of $x \ln(1 + 2x^2)$ at $x = 0$, given that $\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

(c) (4%) Compute $\lim_{x \to 0} \frac{x \ln(1 + 2x^2)}{\int_0^x \sin(t^2)\,dt}$.

Solution:

(a)

$$\int_0^x \sin(t^2)dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (t^2)^{2n+1} dt$$

2 pts for substituting $x = t^2$ in the Taylor series of $\sin x$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^x t^{4n+2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{4n+3} x^{4n+3}$$

4 pts for term-by-term integration

(b)

$$x \cdot \ln(1 + 2x^2) = x \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (2x^2)^n$$

2 pts for substituting $y = 2x^2$ in the Taylor series of $\ln(1 + y)$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 2^n x^{2n+1}$$

2 pts for multiplying x in each term and the final answer

(c)

$$\lim_{x \to 0} \frac{x \ln(1 + 2x^2)}{\int_0^x \sin(t^2)dt} = \lim_{x \to 0} \frac{2x^3 - 2x^5 + \cdots}{\frac{1}{3}x^3 - \frac{1}{3!}\frac{1}{7}x^7 + \cdots} = \lim_{x \to 0} \frac{2 - 2x^2 + \cdots}{\frac{1}{3} - \frac{1}{3!}\frac{1}{7}x^4 + \cdots} = \frac{2}{\frac{1}{3}} = 6$$

2 pts for listing first few terms of Taylor series of $x \ln(1 + 2x^2)$ and $\int_0^x \sin(t^2)dt$.

2 pts for computing the limit as the ratio of coefficients in front of x^3. If students make mistakes in (a) or (b) but they know that the limit is the ratio of x^3's coefficients, they have 2 pts for part (c)