1111模組06-12班 微積分2 期考解答和評分標準

1. Evaluate the following integrals.

   (a) (5%) $\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{2}{3}}}\,\mathrm{d}x$ (b) (6%) $\int \ln(x^2+1)\mathrm{d}x$. (c) (7%) $\int_2^3 \frac{x}{\sqrt{4x-x^2}}\,\mathrm{d}x$

Solution:

(a) Let $x = t^6$ (2 points). So $dx/dt = 6t^5$. The original integral becomes

$$\begin{aligned}\int \frac{1}{t^3+t^4}6t^5dt =& \int \frac{6t^2}{1+t}dt\\ =&6\int t-1+\frac{1}{1+t}dt\\ =&6\Big(\frac{t^2}{2}-t+\ln|1+t|\Big)+C\\ =&3x^{\frac{1}{3}}-6x^{\frac{1}{6}}+6\ln|1+x^{\frac{1}{6}}|+C.\text{(3 points)}\end{aligned}$$

(b) Using integration by parts, we get

$$\begin{aligned}\int \ln(x^2+1)dx =& x\ln(x^2+1)-\int x\frac{2x}{x^2+1}dx \text{ (3 points)}\\ =&x\ln(x^2+1)-\int 2-\frac{2}{x^2+1}dx\\ =&x\ln(x^2+1)-2x+2\tan^{-1}x+C.\text{(3 points)}\end{aligned}$$

(c) First we complete the square

$$\int_2^3 \frac{x}{\sqrt{4x-x^2}}dx = \int_2^3 \frac{x}{\sqrt{4-(x-2)^2}}dx.\text{(2 points)}$$

Then we use the substitution $x-2 = 2\sin\theta$ and $dx/d\theta = 2\cos\theta$ (2 points) . Therefore, the integral becomes

$$\begin{aligned}&\int_0^{\frac{\pi}{6}} \frac{2\sin\theta+2}{2|\cos\theta|}2\cos\theta d\theta \quad \text{(because } 0\le\theta\le\pi/6,\ \cos\theta \text{ is positive)}\\ =&\int_0^{\frac{\pi}{6}} 2\sin\theta+2d\theta\\ =&2(-\cos\theta+\theta)\Big|_0^{\frac{\pi}{6}} = 2-\sqrt{3}+\frac{\pi}{3}.\text{(3 points)}\end{aligned}$$

2. (12%) Let R be the region enclosed by the curve $y = \dfrac{1}{x^2(x^2+2x+2)}$, $1 \le x \le 2$ and the x-axis. Find the volume of the solid obtained by rotating R about the y-axis.

Solution:

1. Set up integral (2 points in total):

By shell's method (1 point), the volume is

$$V = \int_1^2 2\pi x \cdot \frac{1}{x^2(x^2+2x+2)}\,\mathrm{d}x = 2\pi \int_1^2 \frac{1}{x(x^2+2x+2)}\,\mathrm{d}x. \quad \text{(1 point)}$$

2. Partial fraction (4 points in total):

By partial fraction (1 point), we can assume

$$\frac{1}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}. \quad \text{(1 point)}$$

Clear denominators, we have $1 = A(x^2+2x+2) + x(Bx+C)$. (1 point) Thus, by comparing the coefficients, we get $A = \frac{1}{2}$, $B = \frac{-1}{2}$, and $C = -1$. (1 point)

*No matter what method is used, give all 4 points for the part of partial fraction if the final form is correct.

3. Evaluation (6 points in total): Hence, we have

$$\begin{aligned} V &= \pi \int_1^2 \left(\frac{1}{x} - \frac{x+2}{x^2+2x+2}\right)\mathrm{d}x \\ &= \pi\left(\int_1^2 \frac{1}{x}\,\mathrm{d}x - \int_1^2 \frac{x+2}{x^2+2x+2}\,\mathrm{d}x\right). \end{aligned}$$

Note that $\displaystyle\int_1^2 \frac{1}{x}\,\mathrm{d}x = \ln|x|\Big|_1^2 = \ln 2$ (1 point) and

$$\begin{aligned} \int_1^2 \frac{x+2}{x^2+2x+2}\,\mathrm{d}x &= \int_1^2 \frac{x+2}{(x+1)^2+1}\,\mathrm{d}x \quad \text{Let } u = x+1,\ \mathrm{d}u = \mathrm{d}x \\ &= \int_2^3 \frac{u+1}{u^2+1}\,\mathrm{d}u \quad \text{(1 point)} \\ &= \int_2^3 \frac{u}{u^2+1}\,\mathrm{d}u + \int_2^3 \frac{1}{u^2+1}\,\mathrm{d}u \quad \text{Let } v = u^2+1,\ \mathrm{d}v = 2u\,\mathrm{d}u \\ &= \frac{1}{2}\int_5^{10} \frac{\mathrm{d}v}{v} \ \text{(1 point)} \ + \arctan(u)\Big|_2^3 \ \text{(1 pint)} \\ &= \frac{1}{2}\ln|v|\Big|_5^{10} + \arctan 3 - \arctan 2 \\ &= \frac{1}{2}\ln 2 + \arctan 3 - \arctan 2. \ \text{(1 point)} \end{aligned}$$

To sum up,

$$\begin{aligned} V &= \pi\left(\ln 2 - \left(\frac{1}{2}\ln 2 + \arctan 3 - \arctan 2\right)\right) \\ &= \pi\left(\frac{1}{2}\ln 2 - \arctan 3 + \arctan 2\right). \ \text{(1 point)} \end{aligned}$$

3. For $t \neq -1$, consider the function $F(t) = \int_t^{\frac{1-t}{1+t}} \frac{\tan^{-1} x}{1+x} \mathrm{d}x$.

(a) (1%) Evaluate $F(\sqrt{2}-1)$.

(b) (6%) Prove that $F'(t) = \dfrac{A}{1+t}$ with some constant A. Find the constant A.

(**Hint.** You may use, without proof, the fact that $\tan^{-1} t + \tan^{-1}\left(\dfrac{1-t}{1+t}\right) = \dfrac{\pi}{4}$ for $t \neq -1$.)

(c) (4%) Use (a) and (b) to find $F(0)$. Hence evaluate $\int_{\frac{1}{3}}^{\frac{1}{2}} \dfrac{\tan^{-1} x}{1+x} \mathrm{d}x$.

Solution:

(a) For $t = \sqrt{2}-1$, we have $\dfrac{1-t}{1+t} = \dfrac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2}-1$. Hence,

$$F(\sqrt{2}-1) = \int_{\sqrt{2}-1}^{\sqrt{2}-1} \frac{\tan^{-1} x}{1+x} dx = 0 \quad (1\%).$$

(b) By the FTC (1% is allocated for the trial of computing the derivative via FTC, this point is given even if the calculation is incorrect),

$$\begin{aligned} F'(t) &= \frac{\tan^{-1}\left(\frac{1-t}{1+t}\right)}{1+\frac{1-t}{1+t}} \left(\frac{1-t}{1+t}\right)' - \frac{\tan^{-1} t}{1+t} \quad \text{(1\% for the correct application of FTC)} \\ &= \frac{\tan^{-1}\left(\frac{1-t}{1+t}\right)}{1+\frac{1-t}{1+t}} \frac{-2}{(1+t)^2} - \frac{\tan^{-1} t}{1+t} \quad \left(\text{1\% for the correct calculation of } \left(\frac{1-t}{1+t}\right)' = \frac{-2}{(1+t)^2}\right) \\ &= -\frac{\tan^{-1}\left(\frac{1-t}{1+t}\right)}{1+t} - \frac{\tan^{-1} t}{1+t} \quad \text{(1\% for the simplification (trial))} \\ &= -\frac{1}{1+t}\left(\tan^{-1}\left(\frac{1-t}{1+t}\right) + \tan^{-1} t\right) \quad \text{(1\% for the correct simplification)} \\ &= -\frac{\pi}{4} \cdot \frac{1}{1+t} \quad \text{(1\% for the correct answer)}. \end{aligned}$$

(c) By (2), we have

$$F(t) = -\frac{\pi}{4} \cdot \ln|1+t| + C$$

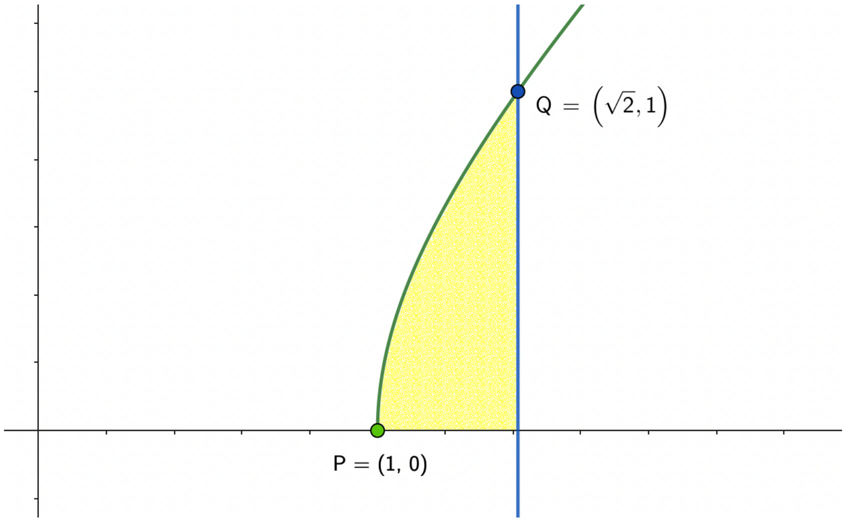
with some constant C (1% for the determination of $F(t)$ up to the constant). By (1),

$$F(\sqrt{2}-1) = -\frac{\pi}{4} \cdot \ln(\sqrt{2}) + C = 0 \quad \text{(1\% for setting up this equation)},$$

so $C = \dfrac{\pi}{8} \cdot \ln 2$. Hence, $F(0) = C = \dfrac{\pi}{8} \cdot \ln 2$ (1% for $F(0)$). Thus,

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\tan^{-1} x}{1+x} dx = F\left(\frac{1}{3}\right) = \frac{\pi}{8} \cdot \ln\frac{9}{8} = \frac{\pi}{8} \cdot (2\ln 3 - 3\ln 2) \quad \text{(1\% for the correct answer)}.$$

4. Let C be the parametric curve defined by $\begin{cases} x(t) = \sec t \\ y(t) = \tan t \end{cases}$, $0 \le t < \frac{\pi}{2}$. Also we let $P = (1, 0)$ and $Q = (\sqrt{2}, 1)$.



(a) (4%) Find the equation of tangent of C at Q.

(b) (3%) Express the arclength of the portion of C from P to Q as an integral. **Do NOT evaluate the integral.**

(c) (7%) Let R be the region bounded by C, the x-axis, and the line $x = \sqrt{2}$. Find the area of R.

Solution:

(a) (1%) The point Q corresponds to $t = \frac{\pi}{4}$.

(1%) $\frac{dx}{dt} = \sec t \tan t$ and $\frac{dy}{dt} = \sec^2 t$.

Therefore, $\frac{dy}{dx} = \underbrace{\frac{dy/dt}{dx/dt}}_{(1\%)} = \frac{\sec t}{\tan t}$.

Hence, the equation of tangent is

$$y - 1 = \frac{\sec \frac{\pi}{4}}{\tan \frac{\pi}{4}}(x - \sqrt{2}) \text{(1\%)} \Rightarrow y - 1 = \sqrt{2}(x - \sqrt{2})$$

Marking scheme for 4a
1% - the value of t that corresponds to Q
1% - finding both $x'(t)$ and $y'(t)$ correctly
1% - formula for dy/dx for a parametric curve
1% - correct equation of tangent line

(b) (1%) The point P corresponds to $t = 0$.

The arclength equals

$$\int_0^{\frac{\pi}{4}} \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}_{(1\%)} = \underbrace{\int_0^{\frac{\pi}{4}} \sqrt{\sec^2 t \tan^2 t + \sec^4 t}\, dt}_{(1\%)}$$

Marking scheme for 4b
1% - the value of t that corresponds to P
1% - correct arclength element ds in parametric form
1% - correct answer (correct integrand AND integration limits)

(c) The area of R equals

$$\int_0^{\frac{\pi}{4}} y(t)\cdot x'(t)dt = \int_0^{\frac{\pi}{4}} \tan t\cdot\sec t\tan t\,\mathrm{d}t\ (1+1\%)$$

$$= \int_0^{\frac{\pi}{4}} \tan^2 t\cdot\sec t\,\mathrm{d}t$$

$$= \int_0^{\frac{\pi}{4}} \sec^3 t - \sec t\,\mathrm{d}t$$

$$= \left[\underbrace{\frac{\ln|\sec t+\tan t|+\sec t\tan t}{2}}_{(3\%)} - \underbrace{\ln|\sec t+\tan t|}_{(1\%)}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\ln(\sqrt{2}+1)+\sqrt{2}}{2} - \ln(\sqrt{2}+1) = \frac{\sqrt{2}-\ln(\sqrt{2}+1)}{2}\ (1\%)$$

Marking scheme for 4c

1% - formula for area : $\int y\,dx$

1% - setting up the correct integral

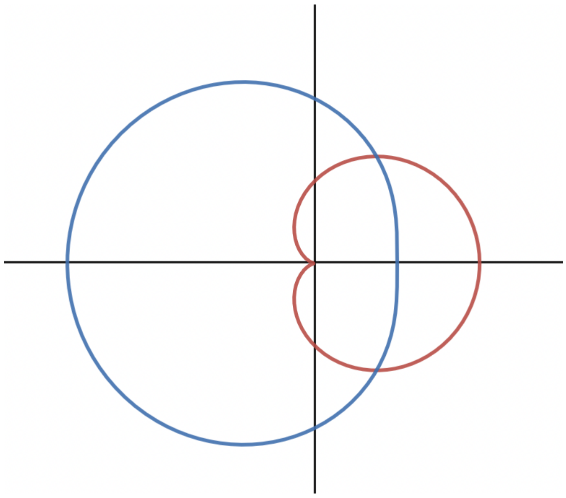
3% - (*) integral of $\sec^3 t$

1% - integral of $\sec t$

1% - answer

Remark for (*). No derivation is required. 1M or 2M can be awarded to a candidate with an incorrect evaluation who (1) attempts reasonably to compute $\int \sec^3\theta d\theta$ or (2) makes minor typos.

5. The following diagram shows the graph of two polar curves $r = 1 + \cos\theta$ and $r = 2 - \cos\theta$.



(a) (4%) Find, in polar coordinates, the intersection points of the two curves.

(b) (1%) Shade clearly in the diagram above the region that lies inside $r = 1 + \cos\theta$ and outside $r = 2 - \cos\theta$.

(c) (6%) Find the area of the region in (b).

Solution:

(a)

Since $r \geq 0$ for both polar curves and both are periodic over 2π, the intersections can only happen when the r values are the same for some θ value.

Set $1 + \cos\theta = 2 - \cos\theta$, then $\cos\theta = \frac{1}{2}$ and $\theta = \pm\frac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$.

When $\theta = \frac{\pi}{3}$, $r = \frac{3}{2}$. When $\theta = -\frac{\pi}{3}$, $r = \frac{3}{2}$.

The intersection points are $\left(\frac{3}{2}, \pm\frac{\pi}{3}\right)_{(r,\theta)}$.

(b)

$2 - \cos\theta \leq 1 + \cos\theta \quad \Rightarrow \quad \cos\theta \geq \frac{1}{2} \quad \Rightarrow \quad -\frac{\pi}{3} + 2k\pi \leq \theta \leq \frac{\pi}{3} + 2k\pi.$

Shade in the right-most region.

(c)

The area is

$$\int_{-\pi/3}^{\pi/3} \frac{1}{2}\ (1 + \cos\theta)^2\ d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2}\ (2 - \cos\theta)^2\ d\theta$$
$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (6\cos\theta - 3)\ d\theta = \frac{3}{2}\left[2\sin\theta - \theta\right]_{-\pi/3}^{\pi/3} = 3\sqrt{3} - \pi$$

□

Grading:

(a) 2% for solving for θ and 2% for final answer. The student does not need to write $2k\pi$ and the final answer could be any equivalent point. Any incorrect answer here still needs to be used in (c).

(b) No partial credit.

(c) 4% for the correct setup (2% for using answer in (a) and 2% for integrand) and 2% for evaluating the integral. An extra -1% if the answer is negative and the student just added absolute value for no reason.

6. An object of mass 1 kg falls near the surface of the earth experiences air resistance that is proportional to the square of its velocity. Therefore, its equation of motion is given by

$$\frac{dv}{dt} = 9.8 - \frac{1}{5}v^2.$$

where $v = v(t)$ is the velocity of the object at time t. It is known that $0 \le v < 7$ and $v(0) = 0$.

(a) (9%) Find $v(t)$.

(b) (1%) Find $\lim_{t\to\infty} v(t)$.

Solution:

(a)

$$\frac{dv}{dt} = \frac{1}{5}(49 - v^2) \Rightarrow \int \frac{dv}{49 - v^2} = \int \frac{1}{5}dt \qquad \text{2 pts}$$

$$\Rightarrow \frac{1}{14}\int \frac{1}{7-v} + \frac{1}{7+v}dv = \frac{1}{5}t + C \qquad \text{2 pts for correct partial fractions}$$

$$\Rightarrow \ln\left|\frac{7+v}{7-v}\right| = 2.8t + C' \qquad \text{2 pts for integrating } \frac{1}{7-v} \text{ and } \frac{1}{7+v}$$

Because $0 \le v < 7$, we conclude that $\frac{7+v}{7-v} = Ae^{2.8t}$, where A is a constant. 1 pt

Because $v(0) = 0$, we have $1 = A \cdot e^0 = A$ 1 pt

Hence $\frac{7+v}{7-v} = e^{2.8t} \Rightarrow v(t) = 7 - \frac{14}{1 + e^{2.8t}} = \frac{7(e^{2.8t} - 1)}{e^{2.8t} + 1}$ 1 pt

(b) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} 7 - \frac{14}{1 + e^{2.8t}} = 7$ 1 pt

7. Initially a tank contains 30 L of pure water. At time t min, brine solution of concentration $c(t) = e^{-\frac{t}{15}}(2 + \sin t)$ kg/L enters the tank at a rate of 2 L/min. The solution is kept mixed thoroughly and drains from the tank at a rate of 2 L/min. Let $A(t)$ (in kg) be the amount of salt in the tank after t minutes.

(a) (4%) Derive a differential equation satisfied by $A(t)$.

(b) (8%) Hence solve for $A(t)$.

Solution:

(a) $\frac{dA}{dt} = \text{rate in} - \text{rate out} = 2 \times e^{-\frac{t}{15}}(2 + \sin t) - 2 \times \frac{A(t)}{30} = 2 \cdot e^{-\frac{t}{15}}(2 + \sin t) - \frac{1}{15}A(t)$

2 pts for rate in $= 2 \times e^{-\frac{t}{15}}(2 + \sin t)$

2 pts for rate out $= 2 \times \frac{A}{30}$

(b) $\frac{dA}{dt} + \frac{1}{15}A(t) = 2 \times e^{-\frac{t}{15}}(2 + \sin t)$

Choose the integrating factor $I(x) = e^{\frac{t}{15}}$ 2 pts

Then $e^{\frac{t}{15}}\left(\frac{dA}{dt} + \frac{1}{15}A(t)\right) = 4 + 2\sin t \Rightarrow \left(e^{\frac{t}{15}} \cdot A(t)\right)' = 4 + 2\sin t$ 2 pts

And $e^{\frac{t}{15}}A(t) = 4t - 2\cos t + C$ 2 pts

Because $A(0) = 0$, we have $e^0 \cdot A(0) = 0 = -2\cos 0 + C$. Hence $C = 2$. 1 pt

Therefore, $A(t) = 4te^{-\frac{t}{15}} - 2e^{-\frac{t}{15}}\cos t + 2e^{-\frac{t}{15}}$ 1 pt

8. Munch-Munch Restaurant in Taipei displays the poster in Figure 1 that indicates every customer should receive their orders within 90 seconds.

Figure 1


Figure 2

It is known that the waiting time for an order is a continuous random variable X whose density is given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ c \cdot 2^{-0.1x} & \text{if } x \geq 0 \end{cases} \quad (x \text{ in seconds}).$$

Recall that $\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\,\mathrm{d}x$.

(a) (3%) Find the value of the constant c.

(b) (3%) A customer receives a gift card as a compensation if his/her order arrives after 90 seconds. Find the probability that a customer will receive a gift card.

(c) In order to shorten the serving time, the manager of the restaurant has purchased a few food serving robots (See Figure 2). Having implemented these robots, the serving time becomes a new random variable $Y = \frac{\sqrt{X}}{2}$.

(i) (3%) Write down the distribution function $F(y) = \mathbb{P}(Y \leq y)$ as an integral.

(ii) (3%) Find the probability density function $f_Y(y)$ of Y. (**Hint.** $f_Y(y) = F'(y)$)

Solution:

(a) (1%) Since $\int_0^\infty c \cdot 2^{-0.1x}\,\mathrm{d}x = 1$,

$$\begin{aligned} \text{LHS} &= \lim_{t\to\infty} \int_0^t c \cdot 2^{-0.1x}\,dx \qquad \text{See below} \\ &= \lim_{t\to\infty} \left[c \cdot \frac{2^{-0.1x}}{-0.1\ln 2} \right]_0^t \text{ (1\%)} \\ &= \lim_{t\to\infty} c \left(\frac{2^{-0.1t}}{-0.1\ln 2} + \frac{1}{0.1\ln 2} \right) \\ &= \frac{c}{0.1\ln 2} \end{aligned}$$

Hence $c = 0.1\ln 2$.(1%)

(b)

$$\begin{aligned} \mathbb{P}(X > 90) = \underbrace{\int_{90}^\infty 0.1\ln 2 \cdot 2^{-0.1x}\,dx}_{(1\%)} &= \underbrace{\lim_{t\to\infty} \int_{90}^t 0.1\ln 2 \cdot 2^{-0.1x}\,dx}_{(1\%)} \\ &= \lim_{t\to\infty} \left[-2^{-0.1x} \right]_{x=90}^{x=t} \\ &= \lim_{t\to\infty} \left(2^{-9} - 2^{-0.1t} \right) \\ &= 2^{-9} \text{ (1\%)} \end{aligned}$$

Marking scheme for 8ab
1% - knowing that the total probability equals 1
1% - anti-derivative of $2^{-0.1x}$
1% - correct value of c
1% - setting up the correct integral for $\mathbb{P}(X > 90)$
1% - (*) definition of improper integral
1% - correct answer
Remark for (*). The definition of improper integrals need to appear <u>at least once</u> in either 8a or 8b. Otherwise, this 1% will be taken off.

(c) (a) For $y \geq 0$ (1%), we have

$$F(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\frac{\sqrt{X}}{2} \leq y) = \underbrace{\mathbb{P}\left(X \leq 4y^2\right)}_{(1\%)} = \underbrace{\int_0^{4y^2} f(x)\,\mathrm{d}x}_{(1\%)}$$

and for $y < 0$, we have $F(y) = 0$.

(b) Let $f_Y(y)$ be the density of Y. For $y \geq 0$ (See above), by FTC, we have

$$f_Y(y) = F'(y) = \underbrace{f(4y^2) \cdot 8y}_{(1\%)} = \underbrace{0.1 \ln 2 \cdot 2^{-0.4y^2} \cdot 8y}_{(2\%)}.$$

and for $y < 0$, we have $f_Y(y) = 0$.

Marking scheme for 8c
1% - (*) distinguish the cases $y > 0$ and $y \leq 0$
1% - transforming $\mathbb{P}(Y \leq y)$ into $\mathbb{P}(X \leq 4y^2)$
1% - correct distribution function (both integrand and integration limits need to be correct)
1% - differentiating $F(y)$ by FTC
2% - correct density $f_Y(y)$ (1% if a candidate obtains incorrect value for c)
Remark for (*). Candidates need to demonstrate the differences of the cases when $y < 0$ and $y \geq 0$ in either (c) (i) or (c) (ii). Otherwise, this 1% will not be awarded.