1111模組01-05班 微積分2 期考解答和評分標準

1. Let $h(u)$ be a continuous function such that $h(u) > 0$ for $u \in \mathbb{R}$. Define

$$g(t) = t\int_t^1 h(u)\,du \quad \text{and} \quad f(x) = \int_0^{x^2} g(t)\,dt.$$

(a) (2%) Find $f'(x)$. Express your answer in terms of h.

(b) (4%) Find the interval(s) on which $f(x)$ is increasing and the interval(s) on which $f(x)$ is decreasing.

(c) (6%) Use integration by parts to write $f(1) = \int_0^1 t\left(\int_t^1 h(u)\,du\right)dt$ as $\int_0^1 p(t)h(t)\,dt$. Find $p(t)$.

Solution:

(a) $f'(x) = \underbrace{2x\cdot g(x^2)}_{1\text{ pt}} = \underbrace{2x^3\int_{x^2}^1 h(u)du}_{1\text{ pt}}$

(b) $x^3 > 0$ for $x > 0$ and $x^3 < 0$ for $x < 0$. 1 pt

$\int_{x^2}^1 h(u)du > 0$ for $x \in (-1,1)$ and $\int_{x^2}^1 h(u)du < 0$ for $x \in (-\infty,-1)\cup(1,\infty)$ 1 pt

Hence $f'(x) > 0$ for $x \in (-\infty,-1)\cup(0,1)$. $f'(x) < 0$ for $x \in (-1,0)\cup(0,1)$.

Therefore $f(x)$ is increasing on $(-\infty,-1)$ and $(0,1)$ 1 pt

$f(x)$ is decreasing on $(-1,0)$ and $(1,\infty)$. 1 pt

(c)

$$f(1) = \int_0^1\left(\int_t^1 h(u)du\right)dt = \int_0^1\left(\int_t^1 h(u)du\right)d\left(\frac{t^2}{2}\right) \qquad 1\text{ pt}$$

$$= \frac{t^2}{2}\left(\int_t^1 h(u)du\right)\Big|_{t=0}^{t=1} - \int_0^1 \frac{t^2}{2}(-h(t))dt \qquad 2\text{ pts}$$

$$= \int_0^1 \frac{t^2}{2}h(t)dt. \qquad 2\text{ pts}$$

Hence $p(t) = \dfrac{t^2}{2}$ 1 pt

1 pt for deciding to integrate t and differentiate $\int_t^1 h(u)du$.

2 pts for applying integration by parts correctly.

2 pts for evaluating $\frac{t^2}{2}\int_t^1 h(u)du$ at $t = 1$ and $t = 0$.

1 pt for $p(t)$

2. (a) (5%) Let $f(x)$ be a continuous function on $[-1,1]$. By using a suitable substitution, show that

$$\int_0^\pi x f(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^\pi f(\sin x)\mathrm{d}x.$$

(b) (7%) The region bounded by the curve $y = \frac{\sin^3 x}{1+\cos^2 x}$ on $0 \le x \le \pi$ and the x-axis is revolved about the y-axis to generate a solid. Use (a) to find the volume of this solid.

Solution:

(a) Apply the substitution $t = \pi - x$ (+4) and $dx = -dt$ (+1)

快速的看一下代入積分後的推導，如下方。 任何小錯 (-1)
(至多扣 1分，不要再多扣分)

$$\int_0^\pi x f(\sin x)\,dx = \int_\pi^0 (\pi - t) f(\sin t)\,(-dt) = \int_0^\pi \pi f(\sin t)\,dt - \int_0^\pi t f(\sin t)\,dt.$$

(b) The method of cylindrical shells gives the volume V as the integral

$$\begin{aligned} V &= 2\pi \int_0^\pi x y(x)\,dx \ (+4) \quad \text{接著使用}(a) \\ &= \pi^2 \int_0^\pi \frac{\sin^3 x}{1+\cos^2 x}\,dx \ (+2) \ \text{下面計算過程不看， 跳至最後檢查答案是否正確} \\ &\overset{u=\cos x}{=} \pi^2 \int_{-1}^1 \frac{1-u^2}{1+u^2}\,du = \pi^2 \int_{-1}^1 \frac{2}{1+u^2} - 1\,du = \pi^2 \left(2\tan^{-1} u - u\right)_{-1}^1 \\ &= \pi^2(\pi - 2).\ (+1) \quad \square \end{aligned}$$

3. Let $f(x)$ be a continuous function on $[1, \infty)$. **Note that $f(x)$ is not necessarily non-negative.**

(a) (4%) Prove that if $\int_1^\infty |f(t)|\, dt$ converges, then $\int_1^\infty f(t)\, dt$ also converges. Hint : consider $g(t) = f(t) + |f(t)|$.

(b) (4%) Determine whether $\int_1^\infty \frac{\cos x}{x^2}\, dx$ is convergent or divergent.

(c) (4%) Determine whether $\int_1^\infty \frac{\sin x}{x}\, dx$ is convergent or divergent. Hint : Use integration by parts.

Solution:

(a) Let $g(t) = f(t) + |f(t)|$.
(1%) Note that $0 \le g(t) \le 2|f(t)|$.
(1%) The convergence of $\int_1^\infty |f(t)|\, dt$ implies that of $\int_1^\infty 2|f(t)|\, dt$. Comparison test implies $\int_1^\infty g(t)\, dt$ is convergent.
(1%) Hence, $\int_1^\infty f(t)\, dt = \int_1^\infty g(t)\, dt - \int_1^\infty |f(t)|\, dt$ is also convergent.
(1%) for overall coherence of the argument

Marking scheme of Question 3a
Since this is a proof-based question, full marks will only be given to a valid argument with no mathematical flaws or ambiguities.

- 1% for the bounds of $g(t)$
- 1% for applying the comparison test to show the convergence of $g(t)$
- 1+1% for completing the argument

(b) Let $f(x) = \frac{\cos x}{x^2}$.
(1%) Since $0 \le |f(x)| \le \frac{1}{x^2}$ and
(1%) $\int_1^\infty \frac{1}{x^2} dx$ is convergent (as a p-integral with $p = 2 > 1$),
(1%) comparison test implies $\int_1^\infty |f(x)| dx$ is convergent. Hence, (a) implies that $\int_1^\infty f(x) dx$ is also convergent.
(1%) for overall coherence of the argument

Marking scheme of Question 3b
Since this is a proof-based question, full marks will only be given to a valid argument with no mathematical flaws or ambiguities.

- 1% for the bounds of $|f(t)|$
- 1% for citing the convergence of p-integral with $p = 2 > 1$
- 1+1% for completing the argument

Remarks.
(1) The following incorrect argument willl receive at most 1% :
$f(x) \le \frac{1}{x^2}$ and the convergence of $\int_1^\infty \frac{1}{x^2} dx$ imply $\int_1^\infty f(x)\, dx$ converges by the comparison test.
(2) No points to candidates who just write down 'convergent' without any reasonable argument.

(c) (1%) By integration by part, we have $\int_1^t \frac{\sin x}{x}\, dx = \frac{\cos t}{t} - \cos 1 + \int_1^t \frac{\cos x}{x^2}\, dx$
(1%) Since $\lim_{t\to\infty} \frac{\cos t}{t} = 0$ (by squeeze theorem) and
(1%) $\lim_{t\to\infty} \int_1^t \frac{\cos x}{x^2}$ is convergent (by (b)), we conclude that $\lim_{t\to\infty} \int_0^t \frac{\sin x}{x}\, dx$ is also convergent.
(1%) for overall coherence of the argument

Marking scheme of Question 3c

Since this is a proof-based question, full marks will only be given to a valid argument with no mathematical flaws or ambiguities.

- 1% for applying IBP to $\int \frac{\sin x}{x}\,dx$ (as a definite/indefinite integral)
- 1% for arguing, correctly, that $\lim_{t\to\infty} \frac{\cos t}{t} = 0$
- 1+1% for completing the argument

4. Consider the initial value problem

$$y'' + 4y = \mathcal{U}(t-3) \text{ with } y(0) = y'(0) = 0.$$

(a) (5%) Let $Y(s)$ be the Laplace transform of $y(t)$. Find $Y(s)$.

(b) (8%) Hence, find $y(t)$. Sketch the graph of $y = y(t)$.

You may use, without proof, the following formulas.

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad \mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}, \quad \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2},$$

$$\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}, \quad \mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0).$$

Solution:

(a) $\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{\mathcal{U}(t-3)\}$

$\Rightarrow s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s}e^{-3s}$

1 pt for $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$

2 pts for $\mathcal{L}\{\mathcal{U}(t-3)\} = \frac{1}{s}e^{-3s}$

$\because y(0) = y'(0) = 0 \therefore Y(s) = \frac{1}{s}e^{-3s}\frac{1}{s^2+4}$

2 pts for plugging in $y(0) = y'(0) = 0$ and solving $Y(s)$

(b) $Y(s) = e^{-3s}\frac{1}{s(s^2+4)} = e^{-3s}(\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4})$

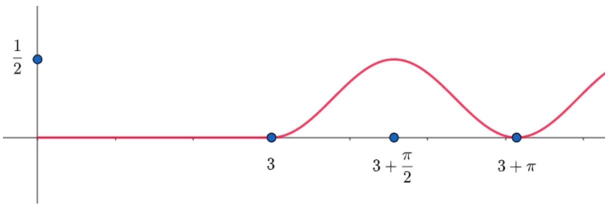
3 pts for correct partial fractions

$y(t) = \frac{1}{4}\mathcal{U}(t-3) - \frac{1}{4}\cos(2(t-3))\mathcal{U}(t-3)$

2 pts for $\mathcal{L}^{-1}\{e^{-3s}\frac{1}{s}\} = \mathcal{U}(t-3)$.

2 pts for $\mathcal{L}^{-1}\{e^{-3s}\frac{s}{s^2+4}\} = \cos(2(t-3))\mathcal{U}(t-3)$

The graph of $y(t)$ is



1 pt for the graph of $y(t)$.

5. In country A, a booster vaccine has been invented to conquer a mutated virus X. Let x (in thousands) be the total number of people in country A at time t (in months) and y be the those who has received the booster shot at time t. It is known that $0 < x < 100$ and x, y satisfies the following equations.

$$(1) \quad \frac{dx}{dt} = 0.2x\left(1 - \frac{x}{100}\right) \qquad (2) \quad \frac{dy}{dt} = 0.4y\left(1 - \frac{y}{x}\right).$$

It is known that $x(0) = 10$ and $y(0) = 1$.

(a) (7%) By solving (1), find x in terms of t. Express your answer in the form $x(t) = \dfrac{100}{f_1(t)}$.

(b) (8%) By letting $u = \dfrac{1}{y}$ in (2), find y in terms of t. Express your answer in the form $y(t) = \dfrac{100}{f_2(t)}$.

(c) (2%) Hence, determine how long will it take for 80% of the population to have received the booster vaccine.

Solution:

(a) (1%) By separating the variables, we have $100\displaystyle\int \frac{1}{x(100-x)}\,dx = \int 0.2dt$

(1%) To compute the LHS, we first note $\dfrac{1}{x(100-x)}dx = \dfrac{1}{100}\left(\dfrac{1}{x} + \dfrac{1}{100-x}\right)$.

(1%) Then $\displaystyle\int \frac{1}{x(100-x)} = \frac{1}{100}\ln\frac{x}{100-x} + (\text{constant})$

(1%) Hence, the equality becomes $\ln\dfrac{x}{100-x} = 0.2t + C$.

(1%) Since $x(0) = 10$, we have $C = \ln\dfrac{1}{9}$.

(2%) Thus, $x(t) = \dfrac{100}{1 + 9e^{-0.2t}}$.

Marking scheme of Question 5a

- 1% for separating the variables correctly
- 1% for correctly decomposing $\dfrac{1}{x(100-x)}$ into partial fractions
- 1% for a correct antiderivative of $\dfrac{1}{x(100-x)}$
- 1% for a correct implicit solution of the equation
- 1% for figuring out the correct constant C
- 2% for the correct answer

(b) Let $y = \dfrac{1}{u}$. Then

(1%) $\dfrac{dy}{dt} = -\dfrac{1}{u^2}\dfrac{du}{dt}$.

(1%) Equation (2) becomes $-\dfrac{1}{u^2}\dfrac{du}{dt} = 0.4\dfrac{1}{u} - \dfrac{0.4}{u^2}\cdot\dfrac{1 + 9e^{-0.2t}}{100}$.

(1%) From this, we obtain a first order linear equation $\dfrac{du}{dt} + 0.4u = 0.004(1 + 9e^{-0.2t})$.

(1%) An integrating factor is $e^{0.4t}$.
Multiplying this to the above equation and integrating it yields

(1%) $u \cdot e^{0.4t} = 0.004\displaystyle\int e^{0.4t} + 9e^{0.2t}\,dt$

(1%) Hence, $u(t) = 0.01 + 0.18e^{-0.2t} + Ce^{-0.4t}$.

(1%) Since, $u(0) = 1/y(0) = 1$, we have $C = 0.81$.

(1%) Thus $y(t) = \dfrac{100}{1 + 18e^{-0.2t} + 81e^{-0.4t}}$.

Marking scheme of Question 5b

- 1% for relating correctly y' and u'
- 1% for transforming the equation in u and t
- 1% for tidying up and obtaining a first order equation in u
- 1% for knowing the method of integrating factor
- 1% for an implicit solution
- 1% for an explicit solution
- 1% for finding out the correct constant C
- 1% for the correct answer

(c) (1%) Set $y/x = 0.8$, we have

$$\frac{1+9e^{-0.2t}}{1+18e^{-0.2t}+81e^{-0.4t}} = \frac{8}{10} \Rightarrow e^{-0.2t} = \frac{1}{36} \text{ or } -\frac{1}{9} \text{ (rejected)}$$

(1%) Hence, $t = 10\ln 6$.

Marking scheme of Question 5c

- 1% for setting $y/x = 0.8$
- 1% for the correct answer

6. (a) (9%) Use the method of undetermined coefficients to find the general solution $y = y(x)$:

$$y'' + 4y = \sin(2x)$$

(b) (9%) Use the method of variation of parameters to find the general solution $y = y(x)$:

$$y'' + y = \csc(x) \text{ with } 0 < x < \frac{\pi}{2}$$

Solution:

(a) The complementary equation $y'' + 4y = 0$ has auxiliary/characteristic equation $r^2 + 4 = 0$ with complex roots $\pm 2i$. Thus it has general solution $y_c = c_1 \cos(2x) + c_2 \sin(2x)$ where c_1, c_2 are arbitrary constants.

[Here from equation $r^2 + 4 = 0$ to roots $\pm 2i$: (+3), finding general solution y_c: (+2).]

Suppose a particular solution is of the form $y_p = ax\cos(2x) + bx\sin(2x)$. Then

$$\sin(2x) = y_p'' + 4y_p = -4a\sin(2x) + 4b\cos(2x).$$

Thus $a = -1/4, b = 0$ and one finds the general solution

$$y = y_p + y_c = -\frac{1}{4}x\cos(2x) + c_1\cos(2x) + c_2\sin(2x).$$

[By default, you have to use the method of undetermined coefficients to claim credit. Here setting correctly $y_p = ax\cos(2x) + bx\sin(2x)$: (+3), solving correctly $y_p = -x\cos(2x)/4$: (+1).]

(b) The complementary equation $y'' + y = 0$ has auxiliary/characteristic equation $r^2 + 1 = 0$ with complex roots $\pm i$. Thus it has general solution $y_c = c_1 \cos x + c_2 \sin x$ where c_1, c_2 are arbitrary constants. We use the method of variation of parameters.

[Here from equation $r^2 + 4 = 0$ to roots $\pm 2i$: (+2), finding general solution y_c: (+1).]

Set

$$y = u_1 \cos x + u_2 \sin x.$$

The method imposes the conditions

$$u_1' \cos x + u_2' \sin x = 0,$$
$$-u_1' \sin x + u_2' \cos x = \csc x,$$

which give

$$u_1' = -1, \quad u_2' = \frac{\cos x}{\sin x}.$$

One finds

$$u_1 = -x + c_1, u_2 = \ln \sin x + c_2.$$

[Here setting correctly the system of equations for u_i': (+2), solving u_i' correctly: (+2), solving correctly u_i: (+2).]

7. (a) (8%) Consider the parametric curve defined by $\begin{cases} x(t) = 3\cos t - \cos(3t) \\ y(t) = 3\sin t - \sin(3t) \end{cases}$, $0 \le t \le \frac{\pi}{2}$. Find the arclength of this curve. (Hint : $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$)

(b) (8%) Consider the portion of the polar curve $C : r = e^{\theta}$ with $0 \le \theta \le \pi$. Let Q be the point on C at which the tangent is vertical (see Figure). Find the area of the region enclosed by the vertical tangent at Q, the x-axis and the curve C.

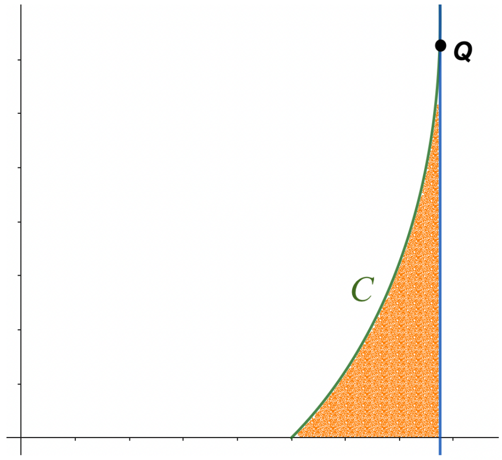


Figure.

Solution:

(a)

$$\begin{aligned} x'(t) &= -3\sin t + 3\sin 3t, \\ y'(t) &= 3\cos t - 3\cos 3t, \\ x'(t)^2 + y'(t)^2 &= 18 - 18\Big(\cos 3t\cos t + \sin 3t \sin t\Big) \quad (1) \\ &= 18\Big(1 - \cos(3t-t)\Big) = 18\Big(1-\cos 2t\Big) \quad (2) \\ &= 18 \cdot 2\sin^2 t = \underline{36}\ (+1)\ \underline{\sin^2 t}.\ (+5) \quad (3) \end{aligned}$$

若 只推導至(1)或(2) (只有 +5)

The arc length is equal to

$$\underline{\int_0^{\pi/2} \sqrt{x'(t)^2 + y'(t)^2}\, dt}\ (+1)$$ 註: 知道求弧長公式，獨立拿 1 分

$$= 6\int_0^{\pi/2} \sin t\, dt = \underline{6}.\ (+1) \quad \square$$

(b) Since $x = e^{\theta}\cos\theta, y = e^{\theta}\sin\theta$,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} = \frac{\sin(\theta + (\pi/4))}{\cos(\theta+(\pi/4))} = \tan(\theta + (\pi/4)).$$

It follows that when θ increases (from 0) to $\pi/4$, one gets a vertical tangent. Or one can verify that when $\theta = \pi/4$ $dx/d\theta = 0$ and $dy/d\theta \ne 0$.

Thus Q has angular coordinate $\underline{\theta = \pi/4}$, or polar coordinate $\underline{(r = e^{\pi/4}, \theta = \pi/4)}$ or rectangular coordinate $\underline{(x = e^{\pi/4}/\sqrt{2}, y = e^{\pi/4}/\sqrt{2})}$ 得到Q 的角度，或任何座標系統的座標 ，若正確 (+3).

First Approach To find the required area, we first compute the area of the polar region enclosed by the curve C (with $\theta \in [0, \pi/4]$), OQ and the x-axis

$$\underline{\frac{1}{2}\int r^2\, d\theta}$$ 知道面積公式 (+1)

$$= \frac{1}{2}\int_0^{\pi/4} e^{2\theta}\, d\theta = \underline{\frac{1}{4}\Big(e^{\pi/2} - 1\Big)}$$ 計算 polar region 面積，若正確 (+3).

The area of the triangle formed by OQ, the tangent, and x-axis is $\frac{1}{2}\left(\frac{e^{\pi/4}}{\sqrt{2}}\right)\cdot\frac{1}{2}\left(\frac{e^{\pi/4}}{\sqrt{2}}\right)=\frac{e^{\pi/2}}{4}$. Hence, the required area equals

$$\frac{e^{\pi/2}}{4}-\frac{1}{4}\left(e^{\pi/2}-1\right)=\underline{\frac{1}{4}}\text{ 直接跳到最後檢查答案，若正確 }(+1).\quad\square$$

Second Approach The polar curve can be treated as a parametric curve, and the required area is the area of the region below the curve and above the x-axis between $x=1$ and $x=e^{\pi/4}/\sqrt{2}$. Thus

$$\begin{aligned}\int y\,dx &= \underline{\int_0^{\pi/4} y(\theta)x'(\theta)\,d\theta}\text{ 知道如此計算 }(+1)\\ &= \int_0^{\pi/4} e^{\theta}\sin\theta\left(e^{\theta}\cos\theta\right)'\,d\theta=\int_0^{\pi/4} e^{2\theta}\sin\theta\left(\cos\theta-\sin\theta\right)d\theta\\ &= \underline{\frac{1}{4}}\text{ 計算過程繁複，或者不看，直接跳到最後檢查答案，若正確 }(+4)\end{aligned}$$

或者查看下列2個積分，每個正確 (+2) 合計4分

Here

$$\begin{aligned}\int_0^{\pi/4} e^{2\theta}\sin\theta\cos\theta\,d\theta &= \frac{1}{2}\int_0^{\pi/4} e^{2\theta}\sin 2\theta\,d\theta\\ &= \frac{1}{8}e^{2\theta}\left(\sin 2\theta-\cos 2\theta\right)\Big|_0^{\pi/4}=\underline{\frac{1}{8}\left(e^{\pi/2}+1\right)}.\end{aligned}$$

$$\int_0^{\pi/4} e^{2\theta}\sin^2\theta\,d\theta=\left[\frac{1}{8}e^{2\theta}+\frac{1}{4}e^{2\theta}\left(\sin^2\theta-\sin\theta\cos\theta\right)\right]_0^{\pi/4}=\underline{\frac{1}{8}\left(e^{\pi/2}-1\right)}.\quad\square$$