1. (15%) Find the limit or show that it doesn't exist.

(a) (6%) $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{|x^2-x|}$ (b) (4%) $\lim_{x\to -\infty} \tan^{-1}\left(\frac{2x^3-x^{\frac{1}{3}}}{x^2+1}\right)$ (c) (5%) $\lim_{x\to 0}\sin(2x)\cot(3x)$

Solution:

(a)

$$\lim_{x\to 0^+} \frac{\sqrt{4+x}-2}{|x^2-x|} = \lim_{x\to 0^+} \frac{\sqrt{4+x}-2}{x-x^2} = \lim_{x\to 0^+} \frac{x}{(x-x^2)(\sqrt{4+x}+2)}$$
$$= \lim_{x\to 0^+} \frac{1}{(1-x)(\sqrt{4+x}+2)} = \frac{1}{4}.(2\%)$$

$$\lim_{x\to 0^-} \frac{\sqrt{4+x}-2}{|x^2-x|} = \lim_{x\to 0^-} \frac{\sqrt{4+x}-2}{x^2-x} = \lim_{x\to 0^-} \frac{x}{(x^2-x)(\sqrt{4+x}+2)}$$
$$= \lim_{x\to 0^+} \frac{1}{(x-1)(\sqrt{4+x}+2)} = -\frac{1}{4}.(2\%)$$

Since $\lim_{x\to 0^+} \frac{\sqrt{4+x}-2}{|x^2-x|} \neq \lim_{x\to 0^-} \frac{\sqrt{4+x}-2}{|x^2-x|}$, we have $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{|x^2-x|}$ does not exist. (2%)

(b)

$$\lim_{x\to -\infty} \tan^{-1}\left(\frac{2x^3-x^{\frac{1}{3}}}{x^2+1}\right) = \tan^{-1}\left(\lim_{x\to -\infty} \frac{2x^3-x^{\frac{1}{3}}}{x^2+1}\right)$$
$$= \tan^{-1}\left(\lim_{x\to -\infty} \frac{2x-x^{-5/3}}{1+\frac{1}{x^2}}\right)$$
$$= \frac{-\pi}{2}.$$

(2 points for computing $\lim_{x\to -\infty} \frac{2x^3-x^{\frac{1}{3}}}{x^2+1} = -\infty$,
2 points for the final answer is $-\frac{\pi}{2}$.

(c) Sol 1:

$\lim_{x\to 0}\sin(2x)\cot(3x) = \lim_{x\to 0}\sin(2x)\frac{\cos(3x)}{\sin(3x)} = \lim_{x\to 0}\frac{\sin(2x)}{\sin(3x)}\cos(3x)$ (1 pt)

$= \lim_{x\to 0}\frac{\sin 2x\cos 3x}{\sin 3x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{2\cos 2x\cos 3x - 3\sin 2x\sin 3x}{3\cos 3x} = \frac{2}{3}$

$\because \lim_{x\to 0}\frac{\sin(2x)}{\sin(3x)} = \lim_{x\to 0}\frac{\sin(2x)}{2x}\cdot\frac{3x}{\sin(3x)}\cdot\frac{2}{3} = \left(\lim_{x\to 0}\frac{\sin(2x)}{2x}\right)\cdot\left(\lim_{x\to 0}\frac{3x}{\sin(3x)}\right)\cdot\frac{2}{3} = \frac{2}{3}$ (3 pts)

and $\lim_{x\to 0}\cos(3x) = 1$

$\therefore \lim_{x\to 0}\sin(2x)\cot(3x) = \lim_{x\to 0}\frac{\sin(2x)}{\sin(3x)}\cos(3x) = \frac{2}{3}\times 1 = \frac{2}{3}$ (1 pt)

Sol 2:

$\lim_{x\to 0}\sin(2x)\cot(3x) = \lim_{x\to 0}\frac{\sin(2x)}{\tan(3x)}$ (1 pt)

$\overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{2\cos(2x)}{3\sec^2(3x)}$ (1 pt for using L'Hospital's Rule. 1 pt for $(\sin 2x)'$. 1 pt for $(\tan(3x))'$.)

$= \frac{2}{3}$ (1 pt)

2. (14%) Consider

$$f(x)=\begin{cases}\dfrac{x^2}{1-e^x} & \text{, if } x\neq 0\\ L & \text{, if } x=0\end{cases}\text{, where } L \text{ is a constant.}$$

Suppose that $f(x)$ is continuous at $x=0$.

(a) (4%) Find the value of the constant L.

(b) (5%) Compute $f'(0)=\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0}$.

(c) (5%) Find $f'(x)$ for $x\neq 0$.

Solution:

(a) First, we compute that

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{x^2}{1-e^x}\underset{\frac{0}{0}(1\%)}{\overset{\text{L.H.}(1\%)}{=}}\lim_{x\to 0}\frac{2x}{-e^x}=0.\ (1\%)$$

Since $f(x)$ is continuous at $x=0$, we have $L=f(0)=\lim_{x\to 0}f(x)=0$ (1%).

(b)

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{\frac{x^2}{1-e^x}-0}{x-0}=\lim_{x\to 0}\frac{x}{1-e^x}\ (1\%)\underset{\frac{0}{0}(1\%)}{\overset{\text{L.H.}(1\%)}{=}}\lim_{x\to 0}\frac{1}{-e^x}=-1.\ (2\%)$$

(c)

$$f'(x)=\left(\frac{x^2}{1-e^x}\right)'=\frac{(x^2)'(1-e^x)-x^2(1-e^x)'}{(1-e^x)^2}\ (3\%)=\frac{2x(1-e^x)+x^2e^x}{(1-e^x)^2}\ (2\%).$$

3. (14%) Find $f'(x)$ of the following functions.

(a) (5%) $f(x) = e^{\tan^{-1}(x^2)}$.

(b) (9%) $f(x) = x^3 \sin(x^2) + x^{\ln x}$.

Solution:

(a) $\frac{d}{dx}(e^{\tan^{-1}(x^2)}) = e^{\tan^{-1}(x^2)} \times (\tan^{-1}(x^2))'$ (2 pts)
$= e^{\tan^{-1}(x^2)} \times \frac{2x}{1+x^4}$ (3 pts)

(b) $(x^3 \cdot \sin(x^2))' = 3x^2 \cdot \sin(x^2) + x^3 \cdot (\sin(x^2))'$ (1 pt for product rule)
$= 3x^2 \cdot \sin(x^2) + x^3 \cdot \cos(x^2) \cdot 2x$ (2 pts for $(\sin(x^2))' = 2x \cdot \cos x^2$)
$= 3x^2 \cdot \sin(x^2) + 2x^4 \cdot \cos(x^2)$
Let $g(x) = x^{\ln x}$.
$\ln(g(x)) = \ln x \cdot \ln x$ (1 pt)
$\xrightarrow{\frac{d}{dx}} \frac{g'(x)}{g(x)} = 2\ln x \cdot \frac{1}{x}$ (2 pts)
Hence $g'(x) = g(x) \cdot 2\ln x \cdot \frac{1}{x} = 2x^{\ln x}\frac{\ln x}{x}$ (2 pts)
Therefore, $f(x) = 3x^2 \sin(x^2) + 2x^4 \cos(x^2) + 2x^{\ln x}\frac{\ln x}{x}$ (1 pt for sum rule)

4. (16%) There is a curve $y^2 = x^3 + 2xy + 7$ on the plane.

(a) (6%) Find $\frac{dy}{dx}$ on the curve.

(b) (4%) Find all points on the curve such that the slope of the tangent line is 1 when $x \geq 0$.

(c) (3%) Find the tangent line of the curve at the point $(1, 4)$.

(d) (3%) The curve is the graph of an implicit function $y = f(x)$ near the point $(1, 4)$. Use the linearization of f at $x = 1$ to estimate $f(1.03)$.

Solution:

(a) Using implicit differentiation, we have that

$$2y\frac{dy}{dx} = 3x^2 + 2y + 2x\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{3x^2 + 2y}{2y - 2x}.$$

(1 points for using implicit differentiation,
3 points for computing $2y\frac{dy}{dx} = 3x^2 + 2y + 2x\frac{dy}{dx}$.
2 points for final answer $\frac{dy}{dx} = \frac{3x^2 + 2y}{2y - 2x}$.)

(b) From $\frac{3x^2 + 2y}{2y - 2x} = 1$, we have $3x^2 = -2x \Rightarrow x = 0$ or $x = \frac{-2}{3}$.
When $x = 0$, $y = \pm\sqrt{7}$.
The slop of tangent line which is 1 are the points $(0, \pm\sqrt{7})$ when $x \geq 0$.
(1 point for $\frac{3x^2 + 2y}{2y - 2x} = 1$.
2 points for the solution is $x = 0$.
1 point for the final answer is $(0, \pm\sqrt{7})$.)

(c) Thus $\frac{dy}{dx}|_{(1,4)} = \frac{11}{6}$. Hence the tangent line is

$$y - 4 = \frac{11}{6}(x - 1).$$

(2 points for $\frac{dy}{dx}|_{(1,4)} = \frac{11}{6}$.
1 point for the tangent line is $y - 4 = \frac{11}{6}(x - 1)$.)

(d) The linearization $L(x) = 4 + \frac{11}{6}(x - 1)$. Thus

$$f(1.03) \approx 4 + \frac{11}{6}(1.03 - 1) \approx 4 + \frac{11}{6} \cdot \frac{3}{100} \approx 4.055.$$

(1 point for the linearization is $L(x) = 4 + \frac{11}{6}(x - 1)$.
2 points for $f(1.03) \approx 4.055$.)

5. (10%) Suppose that the cost function is $C(x) = -30\ln x + 14x + 150$ for $x \geq 1$. Find the absolute minimum of the average cost function $AC(x) = \dfrac{C(x)}{x}$ on $[1, \infty)$. (You need to justify that the answer you find is indeed the absolute minimum.)

Solution:

$AC(x) = \dfrac{C(x)}{x} = -30\dfrac{\ln x}{x} + 14 + \dfrac{150}{x}$ for $x \geq 1$ (2 pts)

$\dfrac{d}{dx}AC(x) = -30\left(\dfrac{1}{x^2} - \dfrac{\ln x}{x^2}\right) - \dfrac{150}{x^2} = \dfrac{-30}{x^2}(6 - \ln x)$

(2 pts for $\left(\dfrac{\ln x}{x}\right)'$. 1 pt for $\left(\dfrac{1}{x}\right)'$)

$\dfrac{d}{dx}AC(x) = 0 \Leftrightarrow x = e^6$ (2 pts)

$\because \dfrac{d}{dx}AC(x) < 0$ for $x \in (1, e^6)$ and $\dfrac{d}{dx}AC(x) > 0$ for $x \in (e^6, \infty)$

$\therefore$ we conclude that $AC(x)$ obtains absolute minimum at $x = e^6$.

(2 pts. If students use $\dfrac{d^2}{dx^2}AC\Big|_{x=e^6} > 0$ to justify that $f(e^6)$ is absolute minimum, they only get 1 pt.)

The absolute minimum is $AC(e^6) = \dfrac{-30}{e^6} + 14$ (1 pt)

6. (17%) Let $f(x) = x^2 \ln x$, for $x > 0$.
   (a) (3%) Find $\lim_{x\to 0^+} f(x)$.
   (b) (5%) Compute $f'(x)$ and find interval(s) of increase and interval(s) of decrease.
   (c) (5%) Compute $f''(x)$ and discuss concavity of $f(x)$.
   (d) (4%) Sketch the graph of $f(x)$. Label the local extremum and inflection point(s) on the curve $y = f(x)$.

Solution:
(a)

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-2}}$$

We observe that

$$\lim_{x\to 0^+} \ln x = -\infty \qquad \text{and} \qquad \lim_{x\to 0^+} x^{-2} = \infty.$$

We can use l'Hospital's Rule for the indeterminate form $\frac{\infty}{\infty}$ to get

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-2}} = \lim_{x\to 0^+} \frac{x^{-1}}{-2x^{-3}}$$

Then

$$\lim_{x\to 0^+} \frac{x^{-1}}{-2x^{-3}} = \lim_{x\to 0^+} -\frac{x^2}{2} = 0$$

□

(b) The first derivative (by product rule)

$$f'(x) = 2x \ln x + x = x(2\ln x + 1)$$

Since we are restricted to $x > 0$, we can solve

$$f'(x) > 0 \quad \Rightarrow \quad 2\ln x + 1 > 0 \quad \Rightarrow \quad x > e^{-1/2}$$

$$f'(x) < 0 \quad \Rightarrow \quad 2\ln x + 1 < 0 \quad \Rightarrow \quad 0 < x < e^{-1/2}$$

The function is increasing on the interval $(e^{-1/2}, \infty)$ and decreasing on the interval $(0, e^{-1/2})$.
For sketching convenience, $e^{-1/2} \approx 0.6$. □

(c) The second derivative (by product rule)

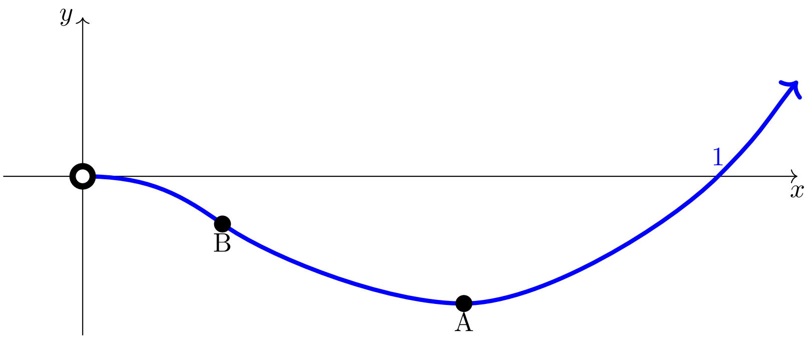
$$f''(x) = 2\ln x + 1 + 2 = 2\ln x + 3$$

$$f''(x) > 0 \quad \Rightarrow \quad 2\ln x + 3 > 0 \quad \Rightarrow \quad x > e^{-3/2}$$

$$f''(x) < 0 \quad \Rightarrow \quad 2\ln x + 3 < 0 \quad \Rightarrow \quad 0 < x < e^{-3/2}$$

The function is concave up on the interval $(e^{-3/2}, \infty)$ and concave down on the interval $(0, e^{-3/2})$.
For sketching convenience, $e^{-3/2} \approx 0.22$. □

(d)
The points we want to label on the graph will be:
A: the local and absolute minimum point $(e^{-1/2}, -1/2e)$
B: the inflection point $(e^{-3/2}, -3/(2e^3))$
Combining all of the above, the sketch



□

Grading:

(a) (3%) 1 point for recognizing the indeterminate form. 2 points for applying l'Hospital's Rule correctly to find the correct answer. Only -1 in the case they forget to say they are using l'H Rule.

(b) (5%) 2 points for correct derivative. 3 points for solving the inequalities. They can get the 3 points even if derivative is wrong.

(c) (5%) 2 points for correct derivative. 3 points for solving the inequalities. They can get the 3 points even if derivative is wrong.

(d) (4%) Check labeled points and the curve connecting them. -1 for each error. This part depends on their answers from (a)-(c). Take points off if the picture doesn't match their answers.

7. (14%) Firm A finds that the total cost $C(x)$ (in dollars) of manufacturing x keyboards/day is given by

$$C(x) = 600 - 50x + 1.8x^2 + 0.04x^3.$$

Each keyboard can be sold at price p dollars related to x by the equation $p(x) = 190 - 3x$. The profit function is $\Pi(x) = x \cdot p(x) - C(x)$.

(a) (8%) Find the daily level of production, x_1, that maximizes the profit $\Pi(x)$.

(b) (3%) The inverse function of $p(x) = 190 - 3x$ is $x = F(p) = \dfrac{190-p}{3}$. Find the point elasticity $\epsilon = \dfrac{F'(p)\cdot p}{F(p)}$.

(c) (3%) (Continued) In the interval $p \in (0, 190)$, find values of p such that $-1 < \epsilon < 0$(inelastic) and values of p such that $\epsilon < -1$(elastic).

Solution:

(a)

$$\begin{aligned} \Pi(x) &= x \cdot p(x) - C(x) = x(190-3x) - (600 - 50x + 1.8x^2 + 0.04x^3) \\ &= -600 + 240x - 4.8x^2 - 0.04x^3 \quad \text{(1 pt)} \\ \Pi'(x) &= 240 - 9.6x - 0.12x^2 \quad \text{(2 pts)} \\ &= -0.12(x^2 + 80x - 2000) = -0.12(x+100)(x-20) \end{aligned}$$

$\Pi'(x) = 0 \Leftrightarrow x = -100$ or $x = 20$ (3 pts for solving $\Pi'(x) = 0$)

However, x can not be negative and $p = 190 - 3x$ can not be negative. Hence $x \in [0, \frac{190}{3}]$.

And $x = 20$ is the only critical point on $(0, \frac{190}{3})$

$\Pi'(x) > 0$ for $x \in (0, 20)$, $\Pi'(x) < 0$ for $x \in (20, \frac{190}{3})$

Hence $\Pi(x)$ obtains absolute maximum at $x = 20$.

(2 pts for justifying $\Pi(20)$ is the absolute maximum.)

($\Pi''(20) < 0$ only shows it's "local" max $\Rightarrow$ -1.)

Or students can compare $\Pi(0) = -600$, $\Pi(\frac{190}{3}) < 0$, $\Pi(20) = 1960$ and conclude that $\Pi(20)$ is the absolute maximum.

(b) $\varepsilon = \dfrac{F'(p)\cdot p}{F(p)} = \dfrac{(-\frac{1}{3}) \times p}{\frac{190-p}{3}} = \dfrac{-p}{190-p}$

(1 pt for $F'(p)$. 2pts for final answer.)

(c) For $p \in (0, 190)$, $190 - p > 0$.

Hence $\varepsilon = \dfrac{-p}{190-p} < -1 \Leftrightarrow -p < -190 + p$ (1 pt)

$\Leftrightarrow 95 < p$ and $p \in (0, 190)$

i.e. $\varepsilon < -1 \Leftrightarrow 95 < p < 190$ (1 pt)

Similarly, $-1 < \varepsilon = \dfrac{-p}{190-p} < 0 \Leftrightarrow p - 190 < -p < 0 \Leftrightarrow 0 < p < 95$ (1 pt)