1111模組01-05班 微積分1 期考解答和評分標準

1. (12%) Evaluate the following limits.

   (a) (6%) $\displaystyle\lim_{x\to-\infty} x\cdot\sin\frac{1}{\sqrt{4x^2+1}}$

   (b) (6%) $\displaystyle\lim_{x\to 0^+}\left(\frac{2^x+3^x+5^x}{3}\right)^{\frac{1}{x}}$

Solution:

(a) $\displaystyle\lim_{x\to-\infty} x\cdot\sin\frac{1}{\sqrt{4x^2+1}} = -\lim_{x\to\infty} x\cdot\sin\frac{1}{\sqrt{4x^2+1}}$ 為 $0\cdot\infty$ 不定型

化為 $\displaystyle = -\lim_{x\to\infty}\frac{x}{\csc\frac{1}{\sqrt{4x^2+1}}}$ $(\frac{\infty}{\infty})$ 或 $\displaystyle = -\lim_{x\to\infty}\frac{\sin\frac{1}{\sqrt{4x^2+1}}}{\frac{1}{x}}$ $(\frac{0}{0})$

方法 (i) Apply L'Hospital's rule, 計算分子、分母導數。關鍵的計算為知道 $\sin\theta$ 或 $\csc\theta$ 的導數。檢查答案 $-\frac{1}{2}$

方法 (ii) 化為 $\displaystyle\left(\lim_{x\to\infty}\frac{x}{\sqrt{4x^2+1}}\right)\left(\lim_{x\to\infty}\frac{\sin\frac{1}{\sqrt{4x^2+1}}}{\frac{1}{\sqrt{4x^2+1}}}\right)$

關鍵：$\displaystyle\lim_{\theta\to0}\frac{\sin\theta}{\theta}=1$，檢查答案 $-\frac{1}{2}$。

(b) $\displaystyle\lim_{x\to0^+}\left(\frac{2^x+3^x+5^x}{3}\right)^{\frac{1}{x}}$ 為 1^∞ 不定型，$\displaystyle=\exp\{\lim_{x\to0^+}\frac{\ln(2^x+3^x+5^x)-\ln 3}{x}\}$ $(\frac{0}{0})$
Apply L'Hospital's rule，計算分子、分母導數。關鍵 $(a^x)', a>0$,
$=(e^{x\ln a})' = \ln a\cdot a^x$, 答案 $(2\cdot3\cdot5)^{1/3}=\sqrt[3]{30}$

2. (10%) Let $f(x)=\ln\left(\dfrac{(x+1)^2}{x^2-x+1}\right)$ and $g(x)=\tan^{-1}\left(\dfrac{2x-1}{\sqrt3}\right)$.

   (a) (8%) Find $f'(x)$ and $g'(x)$.

   (b) (2%) Find two integers m and n such that $\dfrac{d}{dx}(f(x)+\sqrt{m}g(x))=\dfrac{n}{x^3+1}$.

Solution:

(a) $\displaystyle\frac{d}{dx}f=\frac{d}{dx}\ln\left(\frac{(x+1)^2}{x^2-x+1}\right)=\frac{2}{x+1}-\frac{2x-1}{x^2-x+1},\ x\neq-1$
注意：$\ln(x+1)^2=2\ln|x+1|$. 若只寫 $2\ln(x+1)$, 扣 1 分

$\displaystyle\frac{d}{dx}g=\frac{d}{dx}\tan^{-1}\left(\frac{2x-1}{\sqrt3}\right)=\frac{\frac{2}{\sqrt3}}{1+\left(\frac{2x-1}{\sqrt3}\right)^2}=\frac{2\sqrt3}{4x^2-4x+4}$

(b) $\displaystyle\frac{d}{dx}(f+\sqrt{m}g)=\frac{n}{x^3+1}=\frac{-3x+3}{x^3+1}+\frac{\sqrt{m}(\frac{\sqrt3}{2})(x+1)}{x^3+1}$

$\displaystyle\frac{\sqrt3}{2}\sqrt{m}=3,\ 3m=36,\ m=12$

$\displaystyle 3+\sqrt{12}\cdot\frac{\sqrt3}{2}=3+\frac{6}{2}=6=n$

3. (12%) **Figure 1** below shows the curve given by $2(x^2+y^2)^2 = 25(x^2-y^2)$ for $(x,y) \neq (0,0)$.

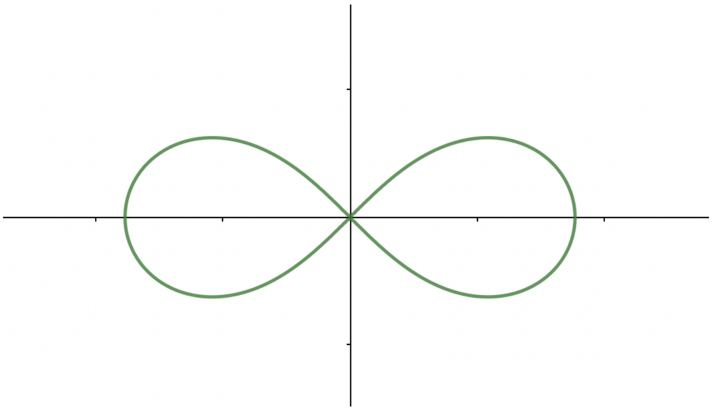

Figure 1.

Find the highest and the lowest points on the curve (that is, the points with, respectively, the largest and the smallest y-coordinates).

Solution:

The curve reaches its highest and lowest points when the derivative y' of y with respect to x is zero. To find y', take the implicit differentiation

$$4(x^2+y^2)(2x+2yy') = 25(2x-2yy'), \tag{1}$$

and one obtains

$$y' = -\frac{x(4x^2+4y^2-25)}{y(4x^2+4y^2+25)}. \tag{2}$$

Thus $y' = 0$ if and only if

$$x^2+y^2 = \frac{25}{4}. \tag{3}$$

In this case, the starting equation gives $x^2-y^2 = 25/8$. Together with (3), one finds $x^2 = 75/16, y^2 = 25/16$. Therefore the curve has horizontal tangents at the four points $(\pm 5\sqrt{3}/4, \pm 5/4)$. One obtains that the highest and the lowest points are respectively

$$\left(\pm\frac{5\sqrt{3}}{4}, \frac{5}{4}\right) \quad \text{and} \quad \left(\pm\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right). \tag{4}$$

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Correct implicit differentiation (1) and (2): +6

Condition for horizontal tangent (3): +3

Correct extreme values (4): +3

4. (16%) Let $f(x) = x + e^{2(x-1)}$.

(a) (2%) Prove that $f(x)$ is a one-to-one function.

(b) Let $g(x) = f^{-1}(x)$ be the inverse function of $f(x)$.

(i) (5%) Find $g(2)$ and $g'(2)$.

(ii) (5%) Prove that $g''(x) < 0$ for all $x \in \mathbb{R}$.

(iii) (4%) Write down the linearization $L(x)$ of $g(x)$ at $x = 2$. Hence determine whether $g(2.1)$ or $L(2.1)$ is larger.

Solution:

(a) (Sol 1):
Because $f'(x) = 1 + 2e^{2(x-1)} > 1 > 0$ for all $x \in \mathbb{R}$, we conclude that $f(x)$ is increasing on $\mathbb{R}$. Hence $f(x)$ is one-to-one.

(Sol 2):
Because $f'(x) = 1 + 2e^{2(x-1)} \neq 0$ for all $x \in \mathbb{R}$, by Rolle's Theorem we know that $f(x)$ is one-to-one.

(1 pt for correct $f'(x)$. 1 pt for applying the increasing test or Rolle's Theorem.)

(b) (i) Since $f(1) = 1 + e^0 = 2$, we know that $f^{-1}(2) = g(2) = 1$. (1 pt)

$f(g(x)) = x \xRightarrow{\frac{d}{dx}} f'(g(x)) \cdot g'(x) = 1$

Therefore, $g'(x) = \dfrac{1}{f'(g(x))}$ (2 pts)

For $x = 2$, $g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(1)}$ (1 pt)

$\because f'(1) = 1 + 2 \cdot e^{2(1-1)} = 1 + 2 \cdot e^0 = 3$

$\therefore g'(2) = \dfrac{1}{3}$ (1 pt)

(ii) $f(g(x)) = x \xRightarrow{\frac{d}{dx}} f'(g(x)) \cdot g'(x) = 1 \xRightarrow{\frac{d}{dx}} f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x) = 0$

Hence $g''(x) = -\dfrac{1}{f'(g(x))} \cdot f''(g(x)) \cdot (g'(x))^2$.

Or differentiate $g'(x) = \dfrac{1}{f'(g(x))}$ and we obtain

$$g''(x) = \frac{d}{dx}\left(\frac{1}{f'(g(x))}\right) = -\frac{f''(g(x)) \cdot g'(x)}{(f'(g(x)))^2} = -\frac{f''(g(x))}{(f'(g(x)))^3}.$$

(1 pt for trying differentiating $f'(g(x)) \cdot g'(x) = 1$ or $g'(x) = \dfrac{1}{f'(g(x))}$. 1 pt for correct differential rule. 1 pt for $g''(x) = -\dfrac{f''(g(x))}{f'(g(x))} \cdot (g'(x))^2$ or $g''(x) = -\dfrac{f''(g(x))}{(f'(g(x)))^3}$.)

$\because f'(x) = 1 + 2e^{2(x-1)} > 0$, $g'(x) = \dfrac{1}{f'(g(x))} > 0$ $f''(x) = 4e^{2(x-1)} > 0$ for all $x \in \mathbb{R}$ (1 pt for f'')

$\therefore g''(x) = -\dfrac{f''(g(x))}{f'(g(x))}(g'(x))^2 = -\dfrac{f''(g(x))}{(f'(g(x)))^3} < 0$ for all x in the domain of $g(x)$. (1 pt for determining $g''(x) < 0$)

(iii) The linearization of $g(x)$ at $x = 2$ is $L(x) = \underbrace{g(2) + g'(2)(x-2)}_{\text{(1 pt)}} = \underbrace{1 + \frac{1}{3}(x-2)}_{\text{(1 pt)}}$

Because $g''(x) < 0$, we know that $g(x)$ is concave downward. Hence the graph of $g(x)$ lies under any tangent line of $y = y(x)$.
Therefore, $g(2.1) < L(2.1)$. (2 pts).

If $\begin{cases} f'' > 0 \\ f' > 0 \\ f, g\text{對稱 } y = x \end{cases} \Rightarrow g'' < 0$ can have 4M.

If $\begin{cases} f'' > 0 \\ f, g\text{對稱 } y = x \end{cases} \Rightarrow g'' < 0$ is incorrect (consider e^{-x}), have 1M from f''.

5. (14%) (a) (6%) Use Mean Value Theorem to prove that for any $0 < a < b$,

$$\frac{\sqrt{b}-\sqrt{a}}{1+b} \le \tan^{-1}(\sqrt{b}) - \tan^{-1}(\sqrt{a}) \le \frac{\sqrt{b}-\sqrt{a}}{1+a}.$$

(b) (8%) Suppose c is a constant such that the limit

$$L = \lim_{x\to\infty} \frac{\tan^{-1}(\sqrt{x^3+1}) - \tan^{-1}(\sqrt{x^3-1})}{x^c} \text{ is non-zero.}$$

Find c and L.

Solution:

(a) **Solution (1) Use usual MVT.**
(1M) $f(x) = \arctan(x)$ is differentiable everywhere,
Take any $0 < x < y$ and apply MVT to f over $[x, y]$, we have

$$\underbrace{\tan^{-1}(y) - \tan^{-1}(x) = \frac{1}{1+c^2}(y-x)}_{(2M)} \text{ for some } \underbrace{x < c < y}_{(1M)}.$$

Since $x < c < y$, we have $\underbrace{\frac{1}{1+y^2} < \frac{1}{1+c^2} < \frac{1}{1+x^2}}_{(1M)}$. Hence, $\frac{y-x}{1+y^2} < \tan^{-1}(y) - \tan^{-1}(x) < \frac{y-x}{1+x^2}$.

Now put $x = \sqrt{a}$ and $y = \sqrt{b}$ (1M), we have

$$\frac{\sqrt{b}-\sqrt{a}}{1+b} < \tan^{-1}(\sqrt{b}) - \tan^{-1}(\sqrt{a}) < \frac{\sqrt{b}-\sqrt{a}}{1+a}.$$

Solution (2) Use Cauchy's MVT.
(1M) $f(x) = \arctan(\sqrt{x})$ and $g(x) = \sqrt{x}$ are differentiable for $x > 0$.
Apply Cauchy's MVT to f and g over $[a, b]$, we have

$$\underbrace{\frac{\tan^{-1}(\sqrt{b}) - \tan^{-1}(\sqrt{a})}{\sqrt{b}-\sqrt{a}} = \frac{f'(c)}{g'(c)}}_{(2M)} \text{ for some } \underbrace{a < c < b}_{(1M)}.$$

Now, $\frac{f'(c)}{g'(c)} = \frac{1}{1+c}$ (1M) and
since $a < c < b$, we have $\frac{1}{1+b} < \frac{1}{1+c} < \frac{1}{1+a}$ (1M).
Hence,

$$\frac{1}{1+b} < \frac{\tan^{-1}(\sqrt{b}) - \tan^{-1}(\sqrt{a})}{\sqrt{b}-\sqrt{a}} < \frac{1}{1+a}$$

and the desired inequality follows.

Marking Scheme for Q5(a).
1M for the hypothesis in applying Mean Value Theorem
2M+1M for the statement of Mean Value Theorem
1M for setting up correct inequalities (given that the derivative needs to be right)
1M for putting $x = \sqrt{a}$ and $y = \sqrt{b}$ (in Sol 1) or for computing $f'(c)/g'(c)$ (in Sol 2).

Remarks.
The desired inequality *cannot* be obtained by applying MVT to $f(x) = \tan^{-1}(\sqrt{x})$. Any such candidates can receive at most 2M.

(b) (1M) Take $a = x^3 - 1$ and $b = x^3 + 1$ in (a), we have

$$\frac{\sqrt{x^3+1}-\sqrt{x^3-1}}{x^3+2} \le \tan^{-1}(\sqrt{x^3+1}) - \tan^{-1}(\sqrt{x^3-1}) \le \frac{\sqrt{x^3+1}-\sqrt{x^3-1}}{x^3}$$

(2M) Rationalize the two ends, we have

$$\frac{2}{(x^3+2)(\sqrt{x^3+1}+\sqrt{x^3-1})} \le \tan^{-1}(\sqrt{x^3+1}) - \tan^{-1}(\sqrt{x^3-1}) \le \frac{2}{x^3(\sqrt{x^3+1}+\sqrt{x^3-1})}$$

(1M) Multiply every term by $x^{4.5}$ (or equivalently divide by $x^{-4.5}$), we have

$$\frac{2x^{4.5}}{(x^3+2)(\sqrt{x^3+1}+\sqrt{x^3-1})} \le \frac{\tan^{-1}(\sqrt{x^3+1}) - \tan^{-1}(\sqrt{x^3-1})}{x^{-4.5}} \le \frac{2x^{4.5}}{x^3(\sqrt{x^3+1}+\sqrt{x^3-1})}$$

Now we compute (2M)

- $\displaystyle\lim_{x\to\infty} \frac{2x^{4.5}}{x^3(\sqrt{x^3+1}+\sqrt{x^3-1})} = \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{1}{x^3}}+\sqrt{1-\frac{1}{x^3}}} = \frac{2}{2} = 1,$
- $\displaystyle\lim_{x\to\infty} \frac{2x^{4.5}}{(x^3+2)(\sqrt{x^3+1}+\sqrt{x^3-1})} = \lim_{x\to\infty} \frac{2}{(1+\frac{2}{x^3})\sqrt{1+\frac{1}{x^3}}+\sqrt{1-\frac{1}{x^3}}} = \frac{2}{1\cdot 2} = 1.$

(1M) By Squeeze Theorem, we have $\displaystyle\lim_{x\to\infty} \frac{\tan^{-1}(\sqrt{x^3+1}) - \tan^{-1}(\sqrt{x^3-1})}{x^{-4.5}} = 1.$

(1M) So $c = -4.5$ and $L = 1$.

Marking Scheme for Q5(b).
1M for applying the inequality in 5(a)
2M for rationalization
1M for choosing an appropriate x^k
2M for computing at least one of limits at the two end correctly, with justification.
1M for applying Squeeze Theorem
1M for correct answers

6. (22%) Consider the function $f(x) = xe^{\frac{1}{x}}$ for $x \neq 0$.

(a) (2%) Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.

(b) (2%) Find all the vertical asymptotes of $y = f(x)$.

(c) (6%) Find the slant asymptote(s) of $y = f(x)$.

(d) (4%) Find $f'(x)$. Write down the interval(s) of increase and interval(s) of decrease of $y = f(x)$.

(e) (4%) Find $f''(x)$. Write down the interval(s) on which $y = f(x)$ is concave upward and the interval(s) on which $y = f(x)$ is concave downward.

(f) (4%) Sketch the graph of $y = f(x)$. Indicate on your sketch (if any) the local extrema, inflection points and asymptotes of the curve.

Solution:

(a) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} xe^{\frac{1}{x}} \overset{\text{let}}{\underset{y=\frac{1}{x}}{=\!=}} \lim_{y\to\infty} \frac{e^y}{y} = \lim_{y\to\infty} \frac{e^y}{1} = \infty$ (1 pt)

As $x \to 0^-$, $\frac{1}{x} \to -\infty$ and $e^{\frac{1}{x}} \to 0$.

Hence $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} xe^{\frac{1}{x}} = \left(\lim_{x\to 0^-} x\right)\left(\lim_{x\to 0^-} e^{\frac{1}{x}}\right) = 0 \times 0 = 0$ (1 pt)

(b) Since $\lim_{x\to 0^+} f(x) = \infty$, $x = 0$ is a vertical asymptote.

For $a \neq 0$, $f(x)$ is continuous at $x = a$ and $\lim_{x\to a} f(x) = a \cdot e^{\frac{1}{a}} \neq \pm\infty$.

Hence $x = a$ is not a vertical asymptote of $y = f(x)$, for $a \neq 0$.

Therefore $x = 0$ is the only vertical asymptote. (1 pt for $x = 0$ is a vertical asymptote. 1 pt for no other vertical asymptotes.)

(c) To find slant asymptotes, we first find their slopes which are

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} e^{\frac{1}{x}} = e^{\lim_{x\to\pm\infty} \frac{1}{x}} = e^0 = 1.$$

(1 pt for trying computing $\lim_{x\to\pm\infty} \frac{f(x)}{x}$. 1 pt for the correct limit.)

Then we compute

$$\lim_{x\to\pm\infty} f(x) - 1\cdot x = \lim_{x\to\pm\infty} x(e^{\frac{1}{x}} - 1) = \lim_{x\to\pm\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} \overset{\text{let}}{\underset{y=\frac{1}{x}}{=\!=}} \lim_{y\to 0^\pm} \frac{e^y - 1}{y} \overset{\frac{0}{0}}{=\!=} \lim_{y\to 0^\pm} \frac{e^y}{1} = 1.$$

(1 pt for trying computing $\lim_{x\to\pm\infty} f(x) - x$. 2 pts for the correct limit.)

Hence $y = x + 1$ is the slant asymptotes. $f(x)$ approaches $x + 1$ both when x approaches ∞ and when x approaches $-\infty$. (1 pt for the final answer.)

(d) $f'(x) = e^{\frac{1}{x}} - \frac{1}{x}e^{\frac{1}{x}}$ (1 pt)

Since $f'(x) = e^{\frac{1}{x}}\frac{(x-1)}{x}$

$f'(x) > 0$ for $x \in (-\infty, 0) \cup (1, \infty)$, $f'(x) < 0$ for $x \in (0, 1)$

Hence $f(x)$ is increasing on $(-\infty, 0) \cup (1, \infty)$ and $f(x)$ is decreasing on $(0, 1)$.

(1 pt for correct partition, $x = 0$, $x = 1$.

1 pt for intervals of increase $(-\infty, 0)$ and $(1, \infty)$.

1 pt for the interval of decrease $(0, 1)$.)

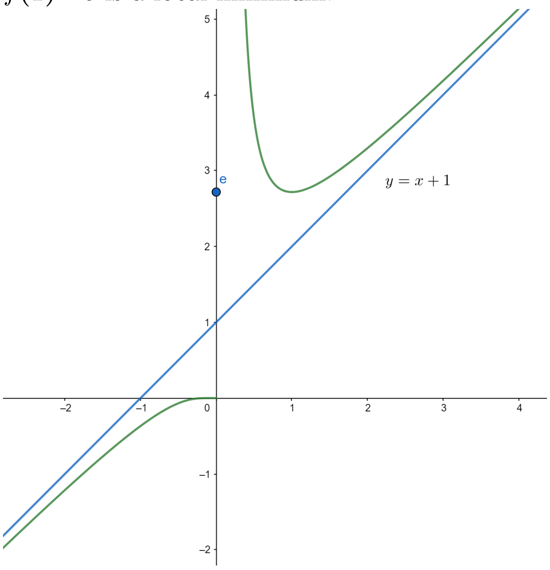
(e) $f'' = -\frac{1}{x^2}e^{\frac{1}{x}} + \frac{1}{x^2}e^{\frac{1}{x}} + \frac{1}{x^3}e^{\frac{1}{x}} = \frac{1}{x^3}e^{\frac{1}{x}}$ (2 pts)

$f''(x) > 0$ for $x \in (0, \infty)$ and $f''(x) < 0$ for $x \in (-\infty, 0)$.

Hence f is concave upward on $(0, \infty)$ and is concave downward on $(-\infty, 0)$.

(1 pt for concavity on $(0, \infty)$. 1 pt for concavity on $(-\infty, 0)$.)

(f) $f(1) = e$ is a local minimum.



(0.5 pt for the local mini $(1, e)$. 0.5 pt for the slant asymptote $y = x + 1$.
1 pt for the graph on $(-\infty, 0)$: increasing and concave downward, $\lim_{x \to 0^-} f(x) = 0$.
1 pt for the graph on $(0, 1)$: decreasing and concave upward, $\lim_{x \to 0^+} f(x) = \infty$.
1 pt for the graph on $(1, \infty)$: increasing and concave upward.)

7. (14%) **Figure 2** below shows a circle C_1 centred at O of radius 1. Let RS be a horizontal chord such that $\angle ROS = 2\theta$ with $0 < \theta < \dfrac{\pi}{2}$ and C_2 be the circle centred at O and tangent to RS. We denote by D the region enclosed by the circles and the chord RS.

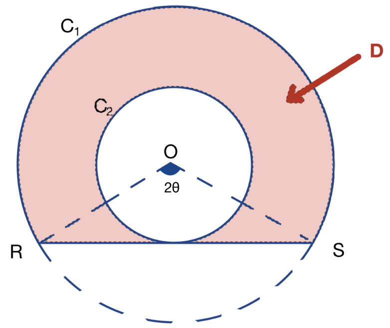


Figure 2.

(a) (8%) Show that the maximum area of the region D equals $\pi - \tan^{-1}(\pi)$ and find the corresponding value of θ at which the maximum value occurs.

(b) (4%) Let s be the perimeter of the region D. Find $\dfrac{ds}{d\theta}$.

(c) (2%) Luke claims that when the area of D is maximized, its perimeter is also maximized. Do you agree with Luke ? Justify.

Solution:

(a) Let A be the area of D.

- (0.5M) Area enclosed by C_1 equals π
- (0.5M) Area enclosed by C_2 equals $\pi\cos^2\theta$
- (0.5M) Area of the triangle ORS equals $\cos\theta\sin\theta$
- (0.5M) Area of the sector ORS equals $\dfrac{2\theta}{2\pi}\cdot\pi = \theta$

Therefore, $A = \pi - \pi\cos^2\theta - \theta + \cos\theta\sin\theta$.

(1M) Then $\dfrac{dA}{d\theta} = 2\pi\sin\theta\cos\theta - 1 + \cos^2\theta - \sin^2\theta = 2\pi\sin\theta\cos\theta - 2\sin^2\theta = 2\sin\theta\cos\theta(\pi - \tan\theta)$.

(1M) Set $\dfrac{dA}{d\theta} = 0$.

(1M) As $0 < \theta < \dfrac{\pi}{2}$, we have $\sin\theta\cos\theta \neq 0$ and hence we have $\tan\theta = \pi \implies \theta = \tan^{-1}(\pi)$ is the only critical number.

(2M) When $0 < \theta < \tan^{-1}(\pi)$, $\dfrac{dA}{d\theta} > 0$ and when $\tan^{-1}(\pi) < \theta < \dfrac{\pi}{2}$, $\dfrac{dA}{d\theta} < 0$, the first derivative test implies that A attains a maximum value when $\theta = \tan^{-1}(\pi)$.

(1M) When $\tan\theta = \pi$, we have $\sin\theta = \dfrac{\pi}{\sqrt{\pi^2+1}}$ and $\cos\theta = \dfrac{1}{\sqrt{\pi^2+1}}$.

Hence, $A(\tan^{-1}\pi) = \pi - \pi\cdot\dfrac{1}{\pi^2+1} - \tan^{-1}(\pi) + \dfrac{\pi}{\pi^2+1} = \pi - \tan^{-1}(\pi)$.

Marking Scheme for Q7(a).
0.5+0.5+0.5+0.5M for writing down the correct area function
($\star$) 1M for derivative of A
1M for setting $dA/d\theta = 0$
($\star$) 1M for finding the critical number $\theta = \tan^{-1}(\pi)$
($\sharp$) 2M for justifying the maximality (accept argument using the second derivative test, or, analysing $\theta \to 0^+$ and $\theta \to (\pi/2)^-$)
($\star$) 1M for correct computation of $A(\tan^{-1}(\pi))$

Remark.
1. Items marked with ($\star$) will only be awarded if the area function is correct.
2. 1M in ($\sharp$) will be given to any candidates who demonstrate ability to justify maximality (for example, using first/second derivative tests), despite having incorrect calculations earlier.

(b)
- (0.5M) Circumference of C_1 equals 2π
- (0.5M) Circumference of C_2 equals $2\pi\cos\theta$
- (0.5M) Perimeter of the arc RS equals 2θ
- (0.5M) Length of the straight line RS equals $2\sin\theta$

Therefore, $s = 2\pi - 2\theta + 2\pi\cos\theta + 2\sin\theta$.

(2M) Hence, $\frac{ds}{d\theta} = -2 - 2\pi\sin\theta + 2\cos\theta$.

Marking Scheme for Q7(b).
$(0.5 + 0.5M) \times 4$ for each of the four terms and its derivative

(c) Since

$$\underbrace{\left.\frac{ds}{d\theta}\right|_{\theta=\tan^{-1}(\pi)}}_{(1M)} = -2 - \frac{2\pi^2}{\sqrt{\pi^2+1}} + \frac{2}{\sqrt{\pi^2+1}} \underbrace{\neq 0}_{(1M)},$$

s does not attain its maximum value at $\theta = \tan^{-1}(\pi)$ so we do not agree with Luke.

Marking Scheme for Q7(c).
1M for evaluating $ds/d\theta$ at the critical point found in (a)
1M for mentioning that the above quality is non-zero and leading to a correct conclusion

Remarks.
1. If a student computed Q7(a) incorrectly or unsuccessfully, he/she can earn at most 1M from this part.
2. Just disagreeing Luke without any valid argument will receive no credits.