1102 Calculus 4 (Applications in Economics and Management) Final Exam Solution

June 4, 2022

There are SIX questions in this examination.

1. (12%) Determine whether each statement is true or false. If true, mark "O". If false, mark "X". You DON'T need to explain your answers.

   (a) (2%) The determinant of a $n \times n$ matrix is unchanged after applying a basic row operation.

   (b) (2%) For a $n \times n$ matrix A, A has full rank if and only if A is invertible.

   (c) (2%) If a matrix A has the reduced echelon form $\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, then the vector $(0,0,1,0,0)$ is in the row space of A.

   (d) (2%) If A is a $n \times n$ positive definite symmetric matrix, then $A + kI_n$ is also positive definite for all $k > 0$.

   (e) (2%) If A is a 2×2 positive definite symmetric matrix, then $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} A \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is also positive definite.

   (f) (2%) If $\mathbf{v_1}$, $\mathbf{v_2}$ are eigenvectors of a $n \times n$ matrix A corresponding to different eigenvalues, then $\mathbf{v_1}$, $\mathbf{v_2}$ are linearly independent.

Solution:

(a) False. The determinant is different by -1 if we exchange any two rows or two columns.

(b) True.

(c) False. The basis of the row space of A is $\{\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}.\}$. It is obvious that $x_1 \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 & 1 & 2 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ which is the same as $x_1 = 0, 2x_1 = 0, x_2 = 1, -x_1 + 2x_2 = 0, x_3 = 0$ has no solution.

(d) True.

(e) False. Let $P = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Then $det(P) = 4 - 4 = 0$. So there exists $x \neq 0$ such that $Px = 0$. Then $x^T P^T A P x = 0$. So $P^T A P$ is not positive definite.

(f) True. Suppose $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. Suppose $\mathbf{v_1}$, $\mathbf{v_2}$ are linearly dependent. We can find $c_1 \neq 0$ and $c_2 \neq 0$ such that $c_1 v_1 + c_2 v_2 = 0$. Then $v_2 = -\frac{c_1}{c_2} v_1$. Since $\lambda_1 \neq \lambda_2$. Thus $Av_1 = \lambda_1 v_1$ and $A(v_2) = A(-\frac{c_1}{c_2} v_1) = -\frac{c_1}{c_2} \lambda_1 v_1 = \lambda_1 v_2$. This contradicts with $Av_2 = \lambda_2 v_2$ with $\lambda_1 \neq \lambda_2$.

2. (14%) Use Sylvester Criterion to determine the definiteness of the following two quadratic forms.

   (a) (7%) $Q(x_1, x_2, x_3) = -x_1^2 - 4x_2^2 - 6x_3^2 + 2x_1x_3 + 4x_2x_3$

   (b) (7%) $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_3 + 4x_2x_3$

Solution:

(a) $Q(x_1, x_2, x_3) = x^T \begin{pmatrix} -1 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -6 \end{pmatrix} x$. Let $A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -6 \end{pmatrix}$.

The leading principle minor of order one is $-1 < 0$.

The leading principle minor of order two is $det\begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} = 4 > 0$.

The leading principle minor of order three is $det\begin{pmatrix} -1 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -6 \end{pmatrix} = -24 - 2 + 4 + 4 = -18 < 0$. So it is negative definite

(b) $Q(x_1, x_2, x_3) = x^T \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix} x$. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$.

The leading principle minor of order one is $1 > 0$.

The leading principle minor of order two is $det\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = 2 > 0$.

The leading principle minor of order three is $det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix} = 4 - 2 - 4 = -2 < 0$. So it is indefinite

3. (20%) Consider the optimization problem :

$$\text{maximize } f(x, y) = \ln(x^2 y) - x - y \text{ subject to } x + y \leq 4,\ x \geq 1,\ y \geq 1.$$

(a) (2%) Explain briefly why the (global) maximum value exists for this problem.

(b) (6%) Write down the classical Lagrangian function and the complete set of first order conditions.

(c) (2%) Verify that NDCQ is satisfied for this problem.

(d) (2%) Show that when $x = 1$, the constraint $x + y \leq 4$ must be binding.

(e) (8%) Find the maximizer and the corresponding maximum value of the optimization problem.

Solution:

(a) The given constraints define a closed and bounded subset on $\mathbb{R}^2$. By Extreme Value Theorem, any continuous functions attain a global maximum on such a subset.

Grading scheme for 3(a)

- 1M for 'closed'
- 1M for 'bounded'

(b) $L(x, y, \lambda_1, \lambda_2, \lambda_3) = \ln(x^2 y) - x - y - \lambda_1(x + y - 4) - \lambda_2(1 - x) - \lambda_3(1 - y)$.

First order conditions :

$$\frac{2}{x} - 1 - \lambda_1 + \lambda_2 = 0, \quad (1)$$

$$\frac{1}{y} - 1 - \lambda_1 + \lambda_3 = 0, \quad (2)$$

$$\lambda_1(x + y - 4) = 0, \quad (3)$$

$$\lambda_2(1 - x) = 0, \quad (4)$$

$$\lambda_3(1 - y) = 0, \quad (5)$$

$$x + y \leq 4, x \geq 1, y \geq 1 \quad (6)$$

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0 \quad (7)$$

Grading scheme for 3(b)

- 2M for correct Lagrangian function
- 0.5M for each of (1), (2), (3), (4), (5), (6)
- 1M for (7)

(c) The 'full' Jacobian matrix is $\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$ which is of rank 2. Now note that at most two constraints can be simutaneously binding so NDCQ is satisfied.

Grading scheme for 3(c)

- 1M for students who demonstrate some understandings about NDCQ (e.g. he/she knows that this means the Jacobian matrix has full rank)
- 1M for any valid argument that NDCQ is valid.

(d) Suppose the constraint $x + y \leq 4$ is not binding. Then (3) implies that $\lambda_1 = 0$. Together with $x = 1$, equation (1) implies $1 + \lambda_2 = 0$ which contradicts with (7), which says $\lambda_2 \geq 0$.

Grading scheme for 3(d)

- All or nothing. These 2Ms are only awarded to a correct (logical) argument.

(e) Suppose $\lambda_2 \neq 0$. Then (4) implies $x = 1$. Using (d), we have $x + y = 4$ and hence $y = 3$. Therefore, (5) implies $\lambda_3 = 0$. But (2) then implies $\lambda_1 = -\frac{2}{3} < 0$ which contradicts with (7).
Therefore, $\lambda_2 = 0$.

Suppose $\lambda_3 \neq 0$. Then (5) implies $y = 1$. Then (2) implies $\lambda_1 = \lambda_3$. Therefore, $\lambda_1 \neq 0$ which implies $x + y = 4$. As $y = 1$, we have $x = 3$. (4) then implies $\lambda_2 = 0$. But (1) then implies $\lambda = -\frac{1}{3}$.
Therefore, $\lambda_3 = 0$.

Having proved that $\lambda_2 = \lambda_3 = 0$, then (2) implies $\lambda_1 = \dfrac{1}{y} - 1$. Since $\lambda_1 \geq 0$, this enforces $y = 1$. This implies $\lambda_1 = 0$. Thus (1) implies $x = 2$. Therefore, we obtain

$$(x, y, \lambda_1, \lambda_2, \lambda_3) = (2, 1, 0, 0, 0).$$

This is the unique solution of FOC and hence is the maximizer for the optimization problme. The corresponding maximum value is $f(2, 1) = \ln 4 - 3$.

Grading scheme for 3(e)

- 2M for a correct proof that $\lambda_2 = 0$.
- 2M for a correct proof that $\lambda_3 = 0$.
- 2M for the solution $(x, y, \lambda_1, \lambda_2, \lambda_3) = (2, 1, 0, 0, 0)$
- 2M for the correct maximum value

4. (18%) The revenue of a company is given by the function $R(x, y, z) = x(yz + 1)$, where x, y, z refers to the units of input in labour, capital and advertising. The manager of the company wants to maximize the revenue but maintaining an upper bound on the total cost, that is to :

$$\text{maximize } R(x, y, z) \text{ subject to } x + y^2 + z^2 \leq 9,\ x \geq 0,\ y \geq 0,\ z \geq 0.$$

(a) (6%) Write down the Kuhn-Tucker's Lagrangian and first order conditions for this optimization problem.

(b) (2%) Check that the Kuhn-Tucker's version NDCQ is satisfied.

(c) (2%) Explain why at any solution to the first order conditions, the constraint $x + y^2 + z^2 \leq 9$ is binding.

(d) (8%) Find the maximizer of the optimization problem.

Solution:

(a) Kuhn-Tucker's Lagrangian : $\tilde{L}(x, y, z, \lambda) = xyz + x - \lambda(x + y^2 + z^2 - 9)$.
Kuhn-Tucker's first order conditions :

$$x(yz + 1 - \lambda) = 0 \quad (1)$$
$$y(xz - \lambda(2y)) = 0 \quad (2)$$
$$z(xy - \lambda(2z)) = 0 \quad (3)$$
$$\lambda(x + y^2 + z^2 - 9) = 0 \quad (4)$$
$$yz + 1 - \lambda \leq 0, \quad xz - \lambda(2y) \leq 0, \quad xy - \lambda(2z) \leq 0 \quad (5)$$
$$x + y^2 + z^2 \leq 9, x \geq 0, y \geq 0, z \geq 0 \quad (6)$$
$$\lambda \geq 0 \quad (7)$$

Grading scheme for 4(a)

- 2M for correct Kuhn-Tucker's Lagrangian function
- 0.5M for each of (1), (2), (3), (4), (6), (7)
- 1M for (5)

(b) Suppose $g(x, y, z) = x + y^2 + z^2 = 9$.

- If $x \neq 0$, then $\frac{\partial g}{\partial x} = 1$ implies any 'reduced' Jacobian matrix would have rank 1 in this case.
- If $x = 0$ but $y, z \neq 0$, then the 'reduced' Jacobian matrix is $(2y, 2z)$ which has rank 1.
- If $x = y = 0$, then $z = 3$ and the 'reduced' Jacobian matrix is (6) which has rank 1.
- If $x = z = 0$, then $y = 3$ and the 'reduced' Jacobian matrix is (6) which has rank 1.
- It is impossible for $x = y = z = 0$.

In all cases, the 'reduced' Jacobian matrix has full rank so Kuhn-Tucker's NDCQ is satisfied.

Grading scheme for 4(b)

- 1M for demonstrating knowledge of what it means by 'Kuhn-Tucker's NDCQ' (which is to check the rank $(\frac{\partial g_i}{\partial x_j})_{ij}$ where g_i comes from binding constraints and $x_j \neq 0$.
- 1M for any correct and complete argument.

(c) By the first inequality in (5), we have $\lambda \geq 1 + yz \geq 1$. So $\lambda \neq 0$ and hence (4) implies $x + y^2 + z^2 = 9$.

Grading scheme for 4(c)

- All or nothing. These 2Ms are only awarded to a correct (logical) argument.

(d) Compare (2) and (3), we have $2\lambda y^2 = 2\lambda z^2$. By (c), we have $\lambda \neq 0$. Therefore, this implies $y = z$.

If $y = z = 0$, then $x + y^2 + z^2 = 9$ implies $x = 9$. Then (1) implies $\lambda = 1$. Therefore, we obtain a solution

$$(x, y, z, \lambda) = (9, 0, 0, 1).$$

If both $y, z \neq 0$, then (2) and (3) becomes $xz = \lambda(2y)$ and $xy = \lambda(2z)$. Since $y = z$ and they are non-zero, we have $x = 2\lambda$.

If $x = 0$, then (2) implies $-\lambda(2y^2) = 0$ and hence $\lambda = 0$. This contradicts with (c). Therefore, $x \neq 0$. Since $x \neq 0$, (1) implies $yz + 1 = \lambda$ and hence $y^2 = z^2 = \lambda - 1$.

Then $x + y^2 + z^2 = 9$ becomes $2\lambda + 2(\lambda - 1) = 9$. Solving gives $\lambda = \frac{11}{4}$. Thus we obtain

$$(x, y, z, \lambda) = \left(\tfrac{11}{2}, \tfrac{\sqrt{7}}{2}, \tfrac{\sqrt{7}}{2}, \tfrac{11}{4}\right)$$

Since $f(9,0,0) = 9$ and $f\left(\frac{11}{2}, \frac{\sqrt{7}}{2}, \frac{\sqrt{7}}{2}\right) = \frac{121}{8} > 9$. Therefore, the maximizer is $\left(\frac{11}{2}, \frac{\sqrt{7}}{2}, \frac{\sqrt{7}}{2}, \frac{11}{4}\right)$.

Grading scheme for 4(d)

- 2M for showing that $y = z$
- 1M for the solution $(9, 0, 0, 1)$
- 2M for showing that $x \neq 0$
- 1M for showing that if $y, z \neq 0$, $x = 2\lambda$ and $y^2 = z^2 = \lambda - 1$
- 1M for the solution $\left(\frac{11}{2}, \frac{\sqrt{7}}{2}, \frac{\sqrt{7}}{2}, \frac{11}{4}\right)$
- 1M for the correct maximizer/maximum value

5. (22%) In a pandemic, a government is planning to subsidize the vaccination for two high-risk populations, x and y doses respectively. It is estimated that this will reduce $f(x, y) = \frac{3}{4}x + \frac{1}{2}y + \frac{1}{32}xy$ number of severe cases. However, due to the financial constraint and the size of a population, x, y must satisfy inequalities $x + y \leq 16$, $y \leq 12$, $x \geq 0$, and $y \geq 0$. How could the policymaker maximize $f(x, y)$, i.e. cut down the number of severe cases?
   (a) (6%) Write down the classical Lagrangian function and the first order conditions for this optimization problem.
   (b) (2%) Verify that NDCQ is satisfied.
   (c) (8%) Find the maximum value of $f(x, y)$ under the given constraints.
   (d) (6%) Suppose that a better vaccine is developed so that $f(x, y)$ is improved to $\frac{3}{4}x + \frac{9}{16}y + \frac{1}{32}xy$ and the new constraints are $x+y \leq 17$, $y \leq 12$, $x \geq 0$, $y \geq 0$. Estimate the maximum value of new $f(x, y)$ under new constraints.

Solution:

(a) $L(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{3}{4}x + \frac{1}{2}y + \frac{1}{32}xy - \lambda_1(x + y - 16) - \lambda_2(y - 12) + \lambda_3 x + \lambda_4 y$.
First order conditions are

$$\begin{cases} \frac{\partial L}{\partial x} = \frac{3}{4} + \frac{1}{32}y - \lambda_1 + \lambda_3 = 0 \quad (1) \\ \frac{\partial L}{\partial y} = \frac{1}{2} + \frac{1}{32}x - \lambda_1 - \lambda_2 + \lambda_4 = 0 \quad (2) \\ \lambda_1(x + y - 16) = 0 \quad (3) \\ \lambda_2(y - 12) = 0 \quad (4) \\ \lambda_3 x = 0 \quad (5) \\ \lambda_4 y = 0 \quad (6) \end{cases} \Rightarrow \begin{cases} \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0 \\ x + y \leq 16 \\ y \leq 12 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

(1 pt for $L(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$, 1 pt for $L_x = 0$, 1 pt for $L_y = 0$, 1 pt for equations (3) (4), 1 pt for equations (5) (6), 1 pt for the rest inequalities.
If students make minor mistakes in computing partial derivatives or list incomplete inequalities, they get

0.5 point out of each 1 point.

If students derive wrong equations (1) and (2) because they have wrong Lagrangian function, they get 0.5 point out of each 1 point.)

(b) Let $g_1(x,y) = x+y$, $g_2(x,y) = y$, $g_3(x,y) = -y$, $g_4(x,y) = -x$. $\vec{\nabla} g_1(x,y) = (1,1)$, $\vec{\nabla} g_2(x,y) = (0,1)$, $\vec{\nabla} g_3(x,y) = (0,-1)$, $\vec{\nabla} g_4(x,y) = (-1,0)$.

The constraints $g_2 \le 12$, $g_3 \le 0$ can not be both binding. And at most two of the constraints $g_1 \le 16$, $g_2 \le 12$, $g_3 \le 0$, $g_4 \le 0$ are simultaneously binding. Moreover, any two of $\vec{\nabla} g_1$, $\vec{\nabla} g_2$, $\vec{\nabla} g_3$, $\vec{\nabla} g_4$ are linearly independent except $\vec{\nabla} g_2$ and $\vec{\nabla} g_3$. Hence we conclude that NDCQ is satisfied.

(1 point for computing $\vec{\nabla} g_1$, $\vec{\nabla} g_2$, $\vec{\nabla} g_3$, $\vec{\nabla} g_4$, 1 point for discussing cases and checking NDCQ.)

(c) To solve these first order conditions, we discuss cases.

case 1: $\lambda_3 > 0, \lambda_4 > 0$

(5),(6) $\Rightarrow x = y = 0$, (3),(4) $\Rightarrow \lambda_1 = \lambda_2 = 0$

But (1) $\Rightarrow \lambda_3 = -\frac{3}{4}$ contradiction!

case 2: $\lambda_3 > 0, \lambda_4 = 0$

(5) $\Rightarrow x = 0$. Since $y \le 12$, $x + y = y < 16$. Thus $\lambda_1 = 0$

(1) $\Rightarrow \frac{3}{4} + \frac{1}{32}y + \lambda_3 = 0 \Rightarrow \lambda_3 < 0$ contradiction!

case 3: $\lambda_3 = 0, \lambda_4 > 0$

(6) $\Rightarrow y = 0$, (1) $\Rightarrow \lambda_1 = \frac{3}{4}$, (3) $\Rightarrow x + y = 16 \Rightarrow x = 16$

(4) $\Rightarrow \lambda_2 = 0$, (2) $\Rightarrow \lambda_4 = -\frac{1}{4}$ contradiction!

case 4: $\lambda_3 = 0 = \lambda_4$

(1) $\Rightarrow \lambda_1 = \frac{3}{4} + \frac{1}{32}y \ge \frac{3}{4} > 0$. (3) $\Rightarrow x + y = 16$.

(4) $\Rightarrow y = 12$ or $\lambda_2 = 0$.

(i) If $y = 12$, then $x = 4$. (1) $\Rightarrow \lambda_1 = \frac{9}{8}$ (2) $\Rightarrow \lambda_2 = \frac{1}{2} + \frac{1}{8} - \frac{9}{8} < 0$ contradiction!

(ii) If $\lambda_2 = 0$, then

(1) $\Rightarrow y = 32\lambda_1 - 24$

(2) $\Rightarrow x = 32\lambda_1 - 16$

$\Rightarrow x = 12,\ y = 4,\ \lambda_1 = \frac{7}{8}$

Solution: $(x^*, y^*, \lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*) = (12, 4, \frac{7}{8}, 0, 0, 0)$

The maximum value is $f(12,4) = 12.5$

(Students may discuss differently according to whether λ_1, λ_2 are zero or positive. Please check whether they discuss complete cases. 6 points for full discussions and students can get partial credits depending on the completeness of their discussions, For example, if they only consider one case out ot four possibilities, they get 1.5 point out of 6 points. 2 points for the final answer.)

(d) Consider the Lagrangian function

$$L(x, y, \vec{\lambda}; a, b) = (\frac{3}{4}x + ay + \frac{1}{32}xy) - \lambda_1(x + y - b) - \lambda_2(y - 12) + \lambda_3 x + \lambda_4 y.$$

Let $(x^*(a,b), y^*(a,b))$ are maximizer with multipliers $\lambda_1^*(a,b)$, $\lambda_2^*(a,b)$, $\lambda_3^*(a,b)$, $\lambda_4^*(a,b)$.

Let $M(a,b) = f(x^*(a,b), y^*(a,b))$

when $a = \frac{1}{2}$, $b = 16$, $(x^*, y^*, \lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*) = (12, 4, \frac{7}{8}, 0, 0, 0)$, $M(\frac{1}{2}, 16) = 12.5$.

Moreover, by the envelope theorem,

$\frac{\partial M}{\partial a}(\frac{1}{2},16) = \frac{\partial L}{\partial a} = y^* = 4$, $\frac{\partial M}{\partial b}(\frac{1}{2},16) = \frac{\partial L}{\partial b} = \lambda_1^* = \frac{7}{8}$.
Hence by linear approximation,

$$M(\frac{9}{16},17) \approx M(\frac{1}{2},16) + \frac{\partial M}{\partial a}(\frac{1}{2},16)(\frac{9}{16}-\frac{1}{2}) + \frac{\partial M}{\partial b}(\frac{1}{2},16)(17-16) = 12.5 + 4(\frac{9}{16}-\frac{1}{2}) + \frac{7}{8}(17-16) = 12.5 + \frac{9}{8}$$

(1 point for knowing envelope theorem, 1.5 point for correct $\frac{\partial L}{\partial a}$, 1.5 point for correct $\frac{\partial L}{\partial b}$, 1 point for linear approximation, 1 pt for final answer.)

6. (14%) Consider $f(x,y,z) = x^2 - 2y^2 - 2z^2 + 4xz$ under constraints $x^2 + y^2 + z^2 = 1$. We find that $(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}})$ together with some μ^* is a critical point of the Lagrangian function

$$L(x,y,z,\mu) = f(x,y,z) - \mu(x^2 + y^2 + z^2 - 1)$$

(a) (4%) Find μ^*.

(b) (4%) Write down the bordered Hessian matrix at $(x,y,z,\mu) = (\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}, \mu^*)$

(c) (6%) Determine whether $f(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}})$ is a local maximum, local minimum or neither on the constraint set.

Solution:

(a) $L(x,y,z,\mu) = x^2 - 2y^2 - 2z^2 + 4xz - \mu(x^2 + y^2 + z^2 - 1)$
$\frac{\partial L}{\partial x} = 2x + 4z - 2x\mu$, $\frac{\partial L}{\partial y} = -4y - 2y\mu$, $\frac{\partial L}{\partial z} = -4z + 4x - 2z\mu$
At $(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}})$ with μ^*, $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0 \Rightarrow \frac{2}{\sqrt{5}} - \frac{8}{\sqrt{5}} - \frac{2}{\sqrt{5}}\mu^* = 0$, $\frac{8}{\sqrt{5}} + \frac{4}{\sqrt{5}} + \frac{4}{\sqrt{5}}\mu^* = 0 \Rightarrow \mu^* = -3$.
(2 points for computing L_x or L_z correctly. 1 point for plugging in $(x,y,z) = (\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}})$. 1 point for solving μ^*.)

(b) $$H(x,y,z,\mu) = \begin{pmatrix} 0 & 2x & 2y & 2z \\ 2x & 2-2\mu & 0 & 4 \\ 2y & 0 & -4-2\mu & 0 \\ 2z & 4 & 0 & -4-2\mu \end{pmatrix}$$

At $(x,y,z,\mu) = (\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}, -3)$, $H = \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & 0 & \frac{-4}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 8 & 0 & 4 \\ 0 & 0 & 2 & 0 \\ \frac{-4}{\sqrt{5}} & 4 & 0 & 2 \end{pmatrix}$

3 points for $H(x,y,z,\mu)$:
(i) 1 point for the first row/column. 0.5 point is deducted if the first row is $(0, -2x, -2y, -2z)$.
(ii) 2 points for L_{xx}, L_{yy}, L_{zz}, L_{xz}, L_{xy}, L_{yz}. 0.5 point is deducted for each wrong partial derivatives.

1 point for $H(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}, -3)$:
If students compute $H(x,y,z,\mu)$ correctly but plug in wrong μ^*, then they get 0.5 point.

(c) Since there are three variables, x,y,z, and one constraint, we need to check the last 2 leading principle minors of $H = \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & 0 & \frac{-4}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 8 & 0 & 4 \\ 0 & 0 & 2 & 0 \\ \frac{-4}{\sqrt{5}} & 4 & 0 & 2 \end{pmatrix}$.

$|H_3| = \det \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix} = -\frac{8}{5}$, $|H_4| = \det H = -80$
$\because$ The last two LPM of H has the same sign with $(-1)^1$
$\therefore f(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}})$ is a local minimum.

2 points for $|H_3|$, 2 points for $|H_4|$.

2 points for correct conclusion from second order conditions. If students have wrong $|H_3|$ or $|H_4|$ but use right reasoning to judge the property of $f(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}})$, they can get 2 points (full credits) for the conclusion part.