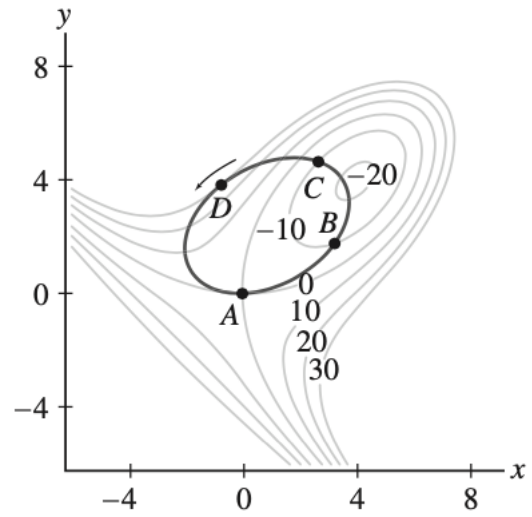
1. (a) (3%) 下圖為函數 $f(x,y)$ 的等高線圖，有一曲線 $\mathbf{c}(t) = (x(t), y(t))$ 沿圖中指示方向行走。請判斷 $\frac{d}{dt}f(\mathbf{c}(t))$ 在 A, B, C 點時是正 (Positive) 、負 (Negative) 還是等於零 (Zero)。
(**提示.** $\frac{d}{dt}f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$)

The following figure shows the level curves of a function $f(x,y)$ and a curve $\mathbf{c}(t) = (x(t), y(t))$ traversed in the direction indicated. Determine whether $\frac{d}{dt}f(\mathbf{c}(t))$ is positive, negative, or zero at A, B, C respectively.
(**Hint.** Recall that $\frac{d}{dt}f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$)



請圈出正確答案 Circle the best answer.

A : Positive / Negative / Zero

B : Positive / Negative / Zero

C : Positive / Negative / Zero

(b) (7%) 考慮由方程 $xz^2 + xy = 1 + y^2 z$ 給定的曲面，求曲面在點 $(1,1,1)$ 上的切面方程式。
Consider a surface defined by the equation $xz^2 + xy = 1 + y^2 z$. Find the equation of the tangent plane of the surface at the point $(1,1,1)$.

Solution:

(a) At points A, the path is tangent to one of the contour lines, so the derivative $\frac{d}{dt}f(c(t))$ is zero. (1%) At point B, the path is moving form a higher contour (-10) to a lower one (-20), so the derivative is negative. (1%) At point C, the path is moving form a higher contour (-10) to a lower one (0), so the derivative is positive. (1%)

(b) Let $F(x,y,z) = xz^2 + xy - 1 - y^2 z$. Then

$$\begin{aligned}\nabla F(1,1,1) &= \left(z^2 + y, x - 2yz, 2xz - y^2\right)\Big|_{(1,1,1)} \quad (3\%)\\ &= (2,-1,1). \quad (3\%)\end{aligned}$$

Hence, the equation of the tangent plane at the point $(1,1,1)$ is equal to

$$2(x-1) - (y-1) + (z-1) = 0. \quad (1\%)$$

2. 考慮函數 Let $f(x,y) = 2x \cdot e^{xy}$.

(a) (4%) 求 Find $\nabla f(2,0)$。

(b) (6%) 用 f 在點 $(2,0)$ 的線性逼近估計 $f(2.06, 0.02)$ 的值。
Use a linear approximation at $(2,0)$ to estimate the value of $f(2.06, 0.02)$.

(c) (6%) 設 Let $g(s,t) = f(2st, s^2 - t^2)$，求 Find $\dfrac{\partial g}{\partial s}(1,1)$ 和 and $\dfrac{\partial g}{\partial t}(1,1)$。

Solution:

(a) $f_x(x,y) = 2e^{xy} + 2xye^{xy} = 2(1+xy)e^{xy}, \quad f_x(2,0) = 2$
$f_y(x,y) = 2x^2e^{xy}, \quad f_y(2,0) = 8$
$\nabla f(2,0) = \langle 2, 8 \rangle$

(b) Linear approximation $L(x,y) = f(2,0) + f_x(2,0)(x-2) + f_y(2,0)(y-0) = 4 + 2(x-2) + 8y$
$L(2.06, 0.02) = 4 + 2(0.06) + 8(0.02) = 4.28$

(c) Let $x(s,t) = 2st,\ y(s,t) = s^2 - t^2$. At $s = 1, t = 1$, we have $x(1,1) = 2,\ y(1,1) = 0$.
We need $\dfrac{\partial x}{\partial s} = 2t$, $\dfrac{\partial x}{\partial t} = 2s$, $\dfrac{\partial y}{\partial s} = 2s$, and $\dfrac{\partial y}{\partial t} = -2t$ for chain rule.
By chain rule:

$$\frac{\partial g}{\partial s}(1,1) = \frac{\partial f}{\partial x}(2,0)\frac{\partial x}{\partial s}(1,1) + \frac{\partial f}{\partial y}(2,0)\frac{\partial y}{\partial s}(1,1) = 2 \cdot 2 + 8 \cdot 2 = 20$$

$$\frac{\partial g}{\partial t}(1,1) = \frac{\partial f}{\partial x}(2,0)\frac{\partial x}{\partial t}(1,1) + \frac{\partial f}{\partial y}(2,0)\frac{\partial y}{\partial t}(1,1) = 2 \cdot 2 + 8 \cdot (-2) = -12$$

Alternative method:
$g(s,t) = 4ste^{2s^3t - 2st^3}$

$$\frac{\partial g}{\partial s} = 4te^{2s^3t-2st^3} + 4st(6s^2t - 2t^3)e^{2s^3t-2st^3}, \quad \frac{\partial g}{\partial s}(1,1) = 4 + 16 = 20$$

$$\frac{\partial g}{\partial t} = 4se^{2s^3t-2st^3} + 4st(2s^3 - 6st^2)e^{2s^3t-2st^3}, \quad \frac{\partial g}{\partial t}(1,1) = 4 - 16 = -12$$

Grading:

(a) 2% for f_x, 2% for f_y, 1% for plugging in (2,0), 1% for knowing the gradient is a vector. This part is only worth 4%, so we take points off for each mistake until there are no points left.

(b) 2% for linear approximation formula, 1% for $f(2,0), f_x(2,0), f_y(2,0)$ each, 3% for plugging in $(2.06, 0.02)$ and the answer. This part is only worth 6%, so we take points off for each mistake until there are no points left.

(c) 3% for each partial derivative. They can use whatever method they prefer. Lose 1% for each small mistake (algebra mistakes) and 3% for each big mistake (wrong formula or poor concept).

3. 設 E 爲一塊佔據區域 $x^2+y^2 \le 4$ 的圓形金屬板。已知在 E 上，點 (x,y) 的溫度爲 $T(x,y)=2y-x^2-2y^2$。

Let E be a circular metal plate that occupies the region $x^2+y^2 \le 4$. The temperature at the point (x,y) on the plate is given by the function $T(x,y)=2y-x^2-2y^2$.

(a) (6%) 求函數 $T(x,y)$ 在 E 的內部的候選點，並判斷它是局部最大值、局部最小值還是鞍點。
Find the critical point(s) of $T(x,y)$ on the interior of E and determine whether it is a local maximum, a local minimum, or a saddle point.

(b) (12%) 使用 Lagrange 乘子法求金屬板邊界 $x^2+y^2=4$ 上的最高和最低溫度。
Use the method of Lagrange multipliers to find the maximum and minimum temperatures on the boundary $x^2+y^2=4$ of the plate.

(c) (4%) 使用 (a) 和 (b)，求金屬板上最熱和最冷的點(包括邊界)。
Using (a) and (b), find the hottest and coldest spots on the plate (which includes the boundary).

Solution:

(a)

Marking scheme.

(2M) For correct derivatives T_x and T_y
(2M) For the correct critical point
(2M) Establishing that it is a maximum with a correct argument

Sample solution.

Set $\underbrace{\begin{cases} T_x = -2x = 0 \\ T_y = 2-4y = 0 \end{cases}}_{(2M)}$ to obtain the only critical point $(x,y) = \underbrace{(0,1/2)}_{(2M)}$. At this point, $D=8>0$ and $A=-2<0$ so this is a (local) maximum point (2M).

(b)

Marking scheme.

(4M) For setting the system of equations (2 marks per equation)
(2M) For dividing into two cases $x=0$ or $\lambda=-1$ (or any reasonable approach)
(4M) For each critical point on the boundary
(2M) For the correct max. and min. values (1 mark each)

Sample solution.

By the method of Lagrange multipliers, we obtain $\begin{cases} -2x = \lambda(2x) \\ 2-4y = \lambda(2y) \end{cases}$ (4M).

The first equation yields $x=0$ or $\lambda=-1$.

- When $x=0$, the constraint $x^2+y^2=4$ gives $y=\pm 2$.
  This gives two crtical points $(0,\pm 2)$ (1+1M).
- When $\lambda=-1$, the second equation becomes $y=1$ and hence the constraint implies $x=\pm\sqrt{3}$. This gives two crtical points $(\pm\sqrt{3},1)$ (1+1M).

Now we compare

$$T(0,2)=-4, \quad T(0,-2)=-12, \quad T(\pm\sqrt{3},1)=-3$$

From this, we conclude that the highest temperature on the boundary is -3 and the lowest temperature is -12.

(c)

Marking scheme.

(2M) Correct method (comparing critical values of (a) and (b))
(2M)* For the coordinates of the hottest spot and of the coldest spot (1 mark each)

Remark. 1M is deducted from * if a candidate writes down the values of extrema rather than the coordinates at which they occur.

Sample solution.
Since the region $x^2 + y^2 \leq 4$ is closed and bounded, the absolute extrema is attained at either the critical point or the boundary (2M).

By (a), $(0, 1/2)$ is the only critical point and at which $T(0, 1/2) = 1/2$. Compare with the findings in (b), we conclude

- the hottest spot is $\underbrace{(0, 1/2)}_{(1M)}$ (at which the temperature is $1/2$),
- the coldest spot is $\underbrace{(0, -2)}_{(1M)}$ (at which the temperature is -12).

4. (a) (10%) 計算 Evaluate $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{2+y^3}\, dy\, dx$.

(b) (10%) 在區域 R 上，函數 $f(x,y)$ 的平均值定義爲
The *average value* of a function $f(x,y)$ over a region R is defined by

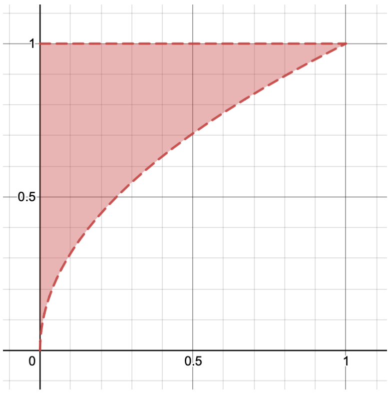
$$\frac{1}{\text{area}(R)} \iint_R f(x,y)\ dA.$$

(其中 area(R) 爲R 的面積)

設 R 爲圓形區域 $x^2+y^2 \le 25$。求 R 上的點到原點的距離的平均值。
Let R be the disk $x^2+y^2 \le 25$. Find the average value of the distance from points on R to the origin.

Solution:

(a) Reverse the order of integration.



$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{2+y^3}\ dy\ dx = \int_0^1 \int_0^{y^2} \sqrt{2+y^3}\ dx\ dy = \int_0^1 y^2\sqrt{2+y^3}\ dy$$

Let $u = 2+y^3$, then $du = 3y^2\ dy$.

$$= \frac{1}{3}\int_2^3 u^{1/2}\ du = \frac{2}{9}\left[u^{3/2}\right]_2^3 = \frac{2(3\sqrt{3}-2\sqrt{2})}{9}$$

(b) The function $f(x,y) = \sqrt{x^2+y^2}$.
Area of R is 25π.

$$\frac{1}{\text{area}(R)} \iint_R f(x,y)\ dA = \frac{1}{25\pi}\int_0^{2\pi}\int_0^5 r^2\ dr\ d\theta = \frac{2}{25}\left[\frac{r^3}{3}\right]_0^5 = \frac{10}{3}$$

Grading:

(a) 6% on setup (2% for knowing to reverse order, 4% for finding the correct bounds), 4% on computation (-1% for algebra and -2% for integration mistakes).

(b) 7% on setup (1% for the area, 2% for knowing to use polar, 2% for finding the correct bounds, 2% for function and $r\ dr\ d\theta$), 3% on computation (-1% for algebra and -2% for integration mistakes).

5. (14%) 設 R 爲曲線 $y=3x^2$、$y=4+3x^2$、$y=8-x^2$ 和 $y=12-x^2$在第一象限中圍成的區域，計算以下的雙重積分：
Let R be the region in the first quadrant bounded by $y=3x^2$, $y=4+3x^2$, $y=8-x^2$ and $y=12-x^2$. Evaluate

$$\iint_R x^3\, dA$$

Solution:

Marking scheme.

(2M) Making a suitable substitution $u=u(x,y)$, $v=v(x,y)$
(2M) Solving for $x=x(u,v)$ and $y=y(u,v)$
(2M) Correct Jacobian
(2M) Correct transformed region
(3M) Correctly transformed the integral (integrand, integration limits)
(1M) Correct anti-derivative for du or dv
(2M) Correct answer

Sample solution.

Let $\begin{cases} u=y-3x^2 \\ v=y+x^2 \end{cases}$ (2M). Therefore, we have $\begin{cases} x=\dfrac{\sqrt{v-u}}{2} \\ y=\dfrac{u+3v}{4} \end{cases}$ (2M). The Jacobian equals

$$J=\begin{vmatrix} -\dfrac{1}{4\sqrt{v-u}} & \dfrac{1}{4\sqrt{v-u}} \\ \dfrac{1}{4} & \dfrac{3}{4} \end{vmatrix} = -\frac{1}{4\sqrt{v-u}} \qquad (2M)$$

The region becomes a rectangle $0\le u\le 4$ and $8\le v\le 12$ (2M).

$$\begin{aligned}\iint_R x^3\,dx &= \int_0^4\int_8^{12}\frac{(v-u)^{3/2}}{8}\cdot\frac{1}{4\sqrt{v-u}}\,dv\,du \qquad (3M)\\ &= \int_0^4\int_8^{12}\frac{v-u}{32}\,dv\,du\\ &= \frac{1}{32}\int_0^4\left[\frac{v^2}{2}-uv\right]_8^{12}du \qquad (1M)\\ &= \frac{1}{32}\int_0^4 40-4u\,du\\ &= 4 \qquad (2M)\end{aligned}$$

6. (18%) 對於下列的微分方程，求滿足指定的初值條件的解 $y = f(t)$。
Solve, for $y = f(t)$, the following differential equations with the given initial conditions.

(a) $ty \cdot \dfrac{dy}{dt} = 1 - y^2, \qquad y(1) = \sqrt{2}.$

(b) $\dfrac{dy}{dt} - 2ty = e^{t^2} \cdot \sin(2t), \qquad y(0) = 1.$

Solution:

(a) Since $y(1) = \sqrt{2}$ (1%), rewrite the equation we have that

$$\frac{y}{1-y^2}\, dy = \frac{1}{t}\, dt. \quad (1\%)$$

Thus

$$-\frac{1}{2}\ln(y^2 - 1) = \int \frac{y}{1-y^2}\, dy = \int \frac{1}{t}\, dt = \ln t + c. \quad (4\%)$$

Put $y(1) = \sqrt{2}$, we get

$$0 = c. \quad (1\%)$$

Hence

$$\ln(y^2 - 1) = -2\ln t = \ln(1/t^2) \Rightarrow y = \sqrt{1 + 1/t^2}. \quad (2\%)$$

(b) Let

$$u(t) := e^{\int (-2t)\, dt} = e^{-t^2} \quad (3\%) \quad \text{(we take the constant to be zero).}$$

Then

$$(e^{-t^2} y)' = \sin(2t). \quad (2\%)$$

Hence

$$e^{-t^2} y = \int \sin(2t)\, dt = -\frac{1}{2}\cos(2t) + c. \quad (2\%)$$

Since $y(0) = 1$, $c = 3/2$. (1%) Therefore,

$$y = e^{t^2}\left(\frac{-1}{2}\cos(2t) + \frac{3}{2}\right). \quad (1\%)$$