1. (12%) Consider the level surface S defined by $x^5 + y^2x^3 - zx = 1$. Near the point $P = (1,1,1)$, the surface defines $x = x(y,z)$ implicitly as a function in y and z.
   (a) (4%) Find the equation of the tangent plane of S at $P = (1,1,1)$.
   (b) (4%) Use implicit function theorem to find $\dfrac{\partial x}{\partial y}$ and $\dfrac{\partial x}{\partial z}$ at $(1,1,1)$.
   (c) (4%) By using a linear approximation at P, approximate the value $x(1.1, 0.9)$.

Solution:

(a) Let $F(x,y,z) = x^5 + y^2x^3 - zx$.

$$\frac{\partial F}{\partial x} = 5x^4 + 3x^2y^2 - z \text{(0.5 point)} \Rightarrow \frac{\partial F}{\partial x}(1,1,1) = 7 \text{(0.5 point)}$$
$$\frac{\partial F}{\partial y} = 2yx^3 \text{(0.5 point)} \Rightarrow \frac{\partial F}{\partial y}(1,1,1) = 2 \text{(0.5 point)}$$
$$\frac{\partial F}{\partial z} = -x \text{(0.5 point)} \Rightarrow \frac{\partial F}{\partial z}(1,1,1) = -1 \text{(0.5 point)}$$

The tangent plane of S at P is $7(x-1) + 2(y-1) - (z-1) = 0$.(1 point)

(b) Use the implicit function theorem, we have that

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x} = -\frac{2yx^3}{5x^4 + 3x^2y^2 - z} \text{(1.5 point)} \Rightarrow \frac{\partial x}{\partial y}(1,1) = -\frac{2}{7} \text{(0.5 point)}$$
$$\frac{\partial x}{\partial z} = -\frac{F_z}{F_x} = \frac{x}{5x^4 + 3x^2y^2 - z} \text{(1.5 point)} \Rightarrow \frac{\partial x}{\partial z}(1,1) = \frac{1}{7} \text{(0.5 point)}$$

(c) The linear approximation at P is

$$x(y,z) - x(1,1) = \frac{\partial x}{\partial y}(1,1)(y-1) + \frac{\partial x}{\partial z}(1,1)(z-1)$$
$$\Rightarrow x(y,z) = 1 - \frac{2}{7}(y-1) + \frac{1}{7}(z-1) \text{(2 points)}.$$

Hence $x(1.1, 0.9) = 1 - \frac{2}{7} \times 0.1 + \frac{1}{7} \times (-0.1) = \frac{67}{70}$(2 points)

2. (10%) Suppose $f(x,y)$ is a differentiable function such that

$$D_{\langle -\frac{3}{5},\frac{4}{5}\rangle} f(0,0) = 2 \text{ and } D_{\langle \frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\rangle} f(0,0) = 2\sqrt{2}.$$

(a) (6%) Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

(b) (4%) At $(0,0)$, find the direction (as a unit vector) in which f increases the most rapidly and the maximum directional derivative at $(0,0)$.

Solution:

(a)

Marking scheme.

(2M)	For knowing $D_{\langle a,b\rangle} f(0,0) = af_x(0,0) + bf_y(0,0)$
(1M+1M)	For setting up the correct system of equation (1 mark per correct equation)
(1M)	For the correct value of $f_x(0,0)$
(1M)	For the correct value of $f_y(0,0)$

Sample solution to 2(a).

Since f is differentiable, we have $\underbrace{D_{\langle a,b\rangle} f(0,0) = af_x(0,0) + bf_y(0,0)}_{(2M)}$.

Therefore, the given derivatives give rise to a system of equations

$$\begin{cases} -\frac{3}{5}f_x(0,0) + \frac{4}{5}f_y(0,0) = 2 & (1M) \\ \frac{1}{\sqrt{2}}f_x(0,0) - \frac{1}{\sqrt{2}}f_y(0,0) = 2\sqrt{2} & (1M) \end{cases} \Rightarrow \begin{cases} -3f_x(0,0) + 4f_y(0,0) = 10 \\ f_x(0,0) - f_y(0,0) = 4 \end{cases}$$

Solving these give $\underbrace{f_x(0,0) = 26}_{(1M)}$ and $\underbrace{f_y(0,0) = 22}_{(1M)}$.

(b)

Marking scheme.

(1M)	For knowing the required direction is $\nabla f(0,0)$
(1M)	For correct direction of steepest ascent (in unit vector)
(1M)	For knowing the max. rate of change is $\lvert\nabla f(0,0)\rvert$
(1M)	For the correct value of $\lvert\nabla f(0,0)\rvert$

Sample solution to 2(b).

The direction in which f increases the most rapidly at $(0,0)$ is $\underbrace{\nabla f(0,0)}_{(1M)} = \langle 26, 22\rangle$ or, in unit vector, is $\underbrace{\langle \frac{26}{\sqrt{1160}}, \frac{22}{\sqrt{1160}}\rangle}_{(1M)}$. And the maximum rate of change equals $\underbrace{|\nabla f(0,0)|}_{(1M)} = \underbrace{\sqrt{1160}}_{(1M)}$.

3. (12%) Find all the critical points of $f(x,y) = 2x^3 + 3x^2y + 2y^3 - 18y$. Determine whether f has a saddle point or a local maximum or minimum at each critical point.

Solution:

$$\begin{aligned} f_x &= 6x^2 + 6xy \\ f_y &= 3x^2 + 6y^2 - 18 \\ f_{xx} &= 12x + 6y \\ f_{xy} &= 6x \\ f_{yy} &= 12y \end{aligned}$$

$$\begin{cases} 6x^2 + 6xy = 0 \\ 3x^2 + 6y^2 - 18 = 0 \end{cases} \Rightarrow \begin{cases} x(x+y) = 0 \\ x^2 + 2y^2 - 6 = 0 \end{cases}$$

Case 1: $x = 0$. It implies that $y^2 = 3 \Rightarrow y = \pm\sqrt{3}$.
Case 2: $x = -y$. It implies that $x^2 = 2 \Rightarrow x = \pm\sqrt{2}$.
Hence the critical points of f are $(0, \pm\sqrt{3}), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$(4 points).

On the other hand, $D(x,y) = \begin{vmatrix} 12x+6y & 6x \\ 6x & 12y \end{vmatrix} = 144xy + 72y^2 - 36x^2$. At $(0,\sqrt{3})$, $D(0,\sqrt{3}) = 216$ and $f_{xx}(0,\sqrt{3}) = 6\sqrt{3} > 0$. By the Second Derivative Test, $f(x,y)$ has a local minimum at $(0,\sqrt{3})$.(2 points)
At $(0,-\sqrt{3})$, $D(0,-\sqrt{3}) = 216$ and $f_{xx}(0,\sqrt{3}) = -6\sqrt{3} < 0$. By the Second Derivative Test, $f(x,y)$ has a local maximum at $(0,-\sqrt{3})$.(2 points)
At $(\sqrt{2},-\sqrt{2}))$, $D(\sqrt{2},-\sqrt{2}) = -216$.By the Second Derivative Test, $f(x,y)$ has a saddle point at $(\sqrt{2},-\sqrt{2})$.(2 points)
At $(-\sqrt{2},\sqrt{2}))$, $D(-\sqrt{2},\sqrt{2}) = -216$.By the Second Derivative Test, $f(x,y)$ has a saddle point at $(-\sqrt{2},\sqrt{2})$.(2 points)

4. (15%) The production of a certain firm can be modelled by the function

$$f(L,K,T) = \sqrt{L} + \sqrt{K} + T$$

where L, K, T denotes units of input of labor, capital and land respectively. The cost in production for this firm is given by the function

$$c(L,K,T) = 4L + 6K + 3T^2.$$

(a) (10%) Suppose the firm maintains a fixed level of production that $f(L,K,T) = 30$. By the method of Lagrange multipliers, find in this case the minimum cost of the firm and the corresponding Lagrange multiplier λ.

(b) (5%) Hence, estimate the minimum cost of the firm when the production level is raised to 30.5. (**Hint.** By linearization, (change in minimum cost) $\approx \lambda \cdot$ (change in production level))

Solution:

(a)

Marking scheme.

(3M)	For correct system of equations arisen from $\nabla f = \lambda \cdot \nabla c$ (1 mark per equation)
(2M)	For attempting to make L, K, T in terms of λ (or any reasonable attempt)
(1+1+1+1M)	For the correct critical point (L, K, T, λ)
(1M)	For the correct minimum value

Sample solution to 4(a).

Set $\nabla c = \lambda \cdot \nabla f$. We have a system of equations $\underbrace{\begin{cases} 4 = \lambda \cdot \dfrac{1}{2\sqrt{L}} \\ 6 = \lambda \cdot \dfrac{1}{2\sqrt{K}} \\ 6T = \lambda \cdot 1 \end{cases}}_{(3M)} \Rightarrow \underbrace{\begin{cases} \sqrt{L} = \dfrac{\lambda}{8} \\ \sqrt{K} = \dfrac{\lambda}{12} \\ T = \dfrac{\lambda}{6} \end{cases}}_{(2M)}$.

Putting these into the constriant, we obtain $\underbrace{\lambda = 80}_{(1M)}$. Hence,

$$\underbrace{L = 100,\ K = \frac{400}{9},\ T = \frac{40}{3}}_{(1+1+1M)}$$

and the minimum cost is $4 \cdot \dfrac{900}{9} + 6 \cdot \dfrac{400}{9} + 3 \cdot \dfrac{1600}{9} = \underbrace{1200}_{(1M)}$.

(Note that this is the absolute minimum value because, for example, $c(0,0,30) = 2700 > 1200$.)

(b)

Marking scheme.

(3M)	For correctly using the hint : 1M for the term 1200, 1M for plugging in λ, 1M for the term $(30.5 - 30)$
(2M)	For the correct answer

Sample solution to 4(b).

Let M_{new} be the new minimum cost under the updated constraint. Then linearizing the minimum cost gives an estimation

$$M_{\text{new}} - \underbrace{1200}_{(1M)} \approx \underbrace{80}_{(1M)} \cdot \underbrace{(30.5 - 30)}_{(1M)}$$

Hence,

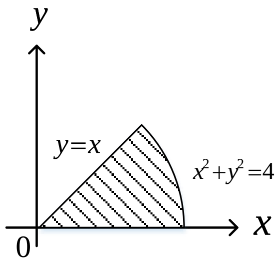
$$M_{\text{new}} \approx 1200 + 80 \cdot (30.5 - 30) = \underbrace{1240}_{(2M)}.$$

5. (14%) Compute the following integrals.

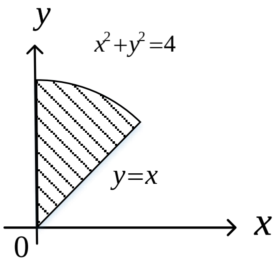
(a) (7%) Choose the region D such that $\int_0^{\sqrt{2}}\int_x^{\sqrt{4-x^2}} \sin(x^2+y^2)dydx = \iint_D \sin(x^2+y^2)dA$.

Compute $\int_0^{\sqrt{2}}\int_x^{\sqrt{4-x^2}} \sin(x^2+y^2)dydx$.

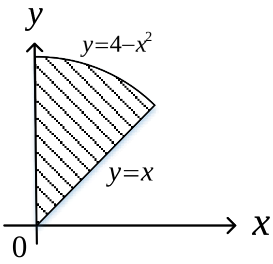
(A)



(B)



(C)



(b) (7%) $\int_0^{\sqrt{\frac{\pi}{2}}}\int_x^{\sqrt{\frac{\pi}{2}}}\int_1^3 \cos(y^2)dzdydx$.

Solution:

(a) (7%) The region D is $\{(x,y)|0 \le x \le \sqrt{2}, x \le y \le \sqrt{4-x^2}\}$. So D is bounded by $y = x, x = 0$, and the circle $x^2+y^2 = 4$. The correct plot is B.

In polar cocodinate, D can be described as $S = \{(r,\theta)|0 \le r \le 2, \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\}$. Using polar coordinate, we have $x^2+y^2 = r^2$, we can integrate it

$$\begin{aligned}
&\int_0^{\sqrt{2}}\int_x^{\sqrt{4-x^2}} \sin(x^2+y^2)dydx \\
=&\int\int_D \sin(x^2+y^2)dA \\
=&\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 \sin(r^2)rdrd\theta \\
=&\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\frac{\cos(r^2)}{2}\Big|_0^2 d\theta \\
=&\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{-\cos(4)+1}{2}d\theta \\
=&\frac{1-\cos(4)}{2}(\frac{\pi}{2}-\frac{\pi}{4}) \\
=&\frac{(1-\cos(4))\pi}{8}
\end{aligned}$$

2 point for the correct expression and the choice of D
1 point the correct expression the region in polar coordinate
1 point the Jacobian factor r in the integration
1 point for getting $\sin(x^2+y^2) = \sin(r^2)$
1 point for getting the $\int_0^2 \sin(r^2)rdr = -\frac{\cos(r^2)}{2}\Big|_0^2$
1 point for the correct answer

(b) (7%) To integrate $\int_0^{\sqrt{\frac{\pi}{2}}}\int_x^{\sqrt{\frac{\pi}{2}}}\int_1^3 \cos(y^2)dzdydx$, we first determine the region of integration $E = \{(x,y,z)|0 \le x \le \sqrt{\frac{\pi}{2}}, x \le y \le \sqrt{\frac{\pi}{2}}, 1 \le z \le 3\} = \{(x,y,z)|0 \le x \le y, 0 \le y \le \sqrt{\frac{\pi}{2}}, 1 \le z \le 3\}$

Thus

$$\begin{aligned}
&\int_0^{\sqrt{\frac{\pi}{2}}}\int_x^{\sqrt{\frac{\pi}{2}}}\int_1^3 \cos(y^2)dzdydx\\
=&\int_0^{\sqrt{\frac{\pi}{2}}}\int_0^{y}\int_1^3 \cos(y^2)dzdxdy\\
=&\int_0^{\sqrt{\frac{\pi}{2}}}\int_0^{y} z\cos(y^2)\Big|_1^3 dxdy\\
=&\int_0^{\sqrt{\frac{\pi}{2}}}\int_0^{y} 2\cos(y^2)dxdy\\
=&\int_0^{\sqrt{\frac{\pi}{2}}} 2y\cos(y^2)dy\\
=&\sin(y^2)\Big|_0^{\sqrt{\frac{\pi}{2}}}\\
=&1
\end{aligned}$$

1 point for the correct expression of the domain of integration $0 \le x \le \sqrt{\frac{\pi}{2}}, x \le y \le \sqrt{\frac{\pi}{2}}, 1 \le z \le 3$

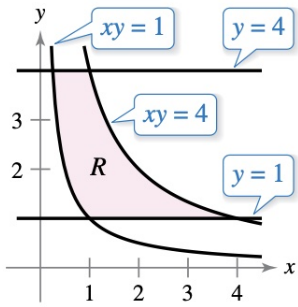
2 point for rewriting the domain of integration $0 \le x \le y, 0 \le y \le \sqrt{\frac{\pi}{2}}, 1 \le z \le 3$

1 point for the integration in z

1 point for the integration in x

2 point for the integration in y

6. (10%) Evaluate the following integral $\iint_R e^{xy} dA$ where R is the region in the following graph.



Solution:

Method 1:

Let $u = xy$ and $v = y$. Then $x = \frac{u}{v}$ and $v = y$. Thus $\frac{\partial x}{\partial u} = \frac{1}{v}, \frac{\partial y}{\partial u} = 0, \frac{\partial x}{\partial v} = -\frac{u}{v^2}, \frac{\partial y}{\partial v} = 1$. The Jacobian matrix

$\frac{\partial(x,y)}{\partial(u,v)} = det \begin{bmatrix} \frac{1}{v} & 0 \\ -\frac{u}{v^2} & 1 \end{bmatrix} = \frac{1}{v}$

The region of integration in $u-v$ coordinate is $S = \{(u,v)|.0 \leq u \leq 4, 0 \leq v \leq 4\}$.

Note that the Jacobian $\frac{1}{v} > 0$ in S. Thus

$$\begin{aligned}
&\int\int_R e^{xy} dA \\
&= \int\int_S e^u \cdot |\frac{1}{v}| dudv \\
&= \int_1^4 \int_1^4 \frac{e^u}{v} dudv \\
&= \int_1^4 \frac{1}{v} dv \int_1^4 e^u du \\
&= \ln 4 \cdot (e^4 - e)
\end{aligned}$$

1 point for the correct expression of the $u = xy$ and $v = y$, 1 point for $x = \frac{u}{v}$ and $v = y$

2 point the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$

1 point for the the correct expression for the region of integration in $u-v$ coordinate

1 point for the correct integration expression $\frac{e^u}{v} dudv$

2 point for the integration in u

2 point for the integration in v

Method 2:

$R = \{(x.y)|1 \leq xy \leq 4, 1 \leq y \leq 4\} = \{(x.y)|\frac{1}{y} \leq x \leq \frac{4}{y}, 1 \leq y \leq 4\}$.

Thus

$$\begin{aligned}
&\int\int_R e^{xy} dA \\
&= \int_1^4 \int_{\frac{1}{y}}^{\frac{4}{y}} e^{xy} dxdy \\
&= \int_1^4 \frac{e^{xy}}{y}\Big|_{\frac{1}{y}}^{\frac{4}{y}} dy \\
&= \int_1^4 \frac{e^4 - e}{y} dy \\
&= \ln 4 \cdot (e^4 - e)
\end{aligned}$$

2 point for the correct expression $R = \{(x.y)|\frac{1}{y} \leq x \leq \frac{4}{y}, 1 \leq y \leq 4\}$.

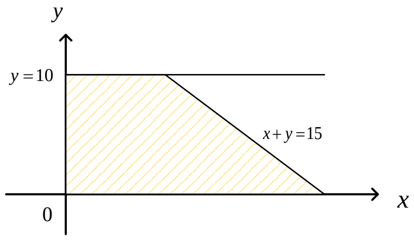
2 point for the expression $\int_{1}^{4}\int_{\frac{1}{y}}^{\frac{4}{y}} e^{xy}dxdy$

3 points for the the correct integration of $\int_{\frac{1}{y}}^{\frac{4}{y}} e^{xy}dx = \frac{e^4 - e}{y}$

3 points for the correct integration of $\int_{1}^{4} \frac{e^4 - e}{y}dy = \ln 4 \cdot (e^4 - e)$

7. (16%) Suppose that in a movie theater the waiting time X for the ticket purchase and the waiting time Y for buying popcorn have the joint probability density function $f(x,y) = \begin{cases} \frac{1}{50}e^{-\frac{x}{5}} & \text{, if } 0 \le y \le 10 \text{ and } x \ge 0. \\ 0 & \text{, otherwise.} \end{cases}$

(a) (6%) Find the probability that a customer waits a total of less than 15 minutes, $P(X+Y \le 15)$.



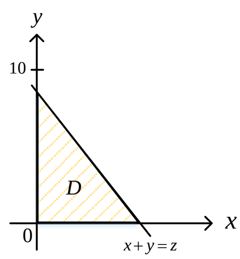
Solution:
$P(X+Y \le 15) = P((x,y) \in D) = \iint_D f(x,y)\,dA$, where D is the region bounded by $x=0$, $y=0$, $y=10$ and $x+y=15$.

$$\begin{aligned}\text{Hence } P(X+Y\le 15) &= \int_0^{10}\int_0^{15-y} \frac{1}{50}e^{-\frac{x}{5}}\,dx\,dy \text{(2 pts for correct form of iterated integral)}\\ &= \int_0^{10}\left(-\frac{1}{10}e^{-\frac{x}{5}}\Big|_{x=0}^{x=15-y}\right)dy\\ &= \int_0^{10} -\frac{1}{10}(e^{-3}\cdot e^{\frac{y}{5}}-1)dy \text{(2 pts for integration w.r.t. } x)\\ &= \left(\frac{-e^{-3}}{2}e^{\frac{y}{5}}+\frac{y}{10}\right)\Big|_{y=0}^{y=10} = -\frac{1}{2e}+1+\frac{1}{2e^3}\end{aligned}$$

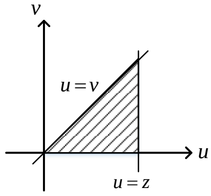
(2 pts for integration w.r.t. y and the final answer.)

If students express $P(X+Y \le 15)$ as a wrong iterated integral and do integration correctly, they get 1 pt for the part of integration.

(b) (6%) For some $0 < z < 10$, the probability of $P(X+Y \le z)$ is $\iint_D f(x,y)dA$, where D is the region shown in the figure. By the change of variables $\begin{cases} u = x+y \\ v = x \end{cases}$, find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ and hence write $\iint_D f(x,y)dA$ as $\int_0^z \int_{h_1(u)}^{h_2(u)} g(u,v)dvdu$.



Solution:
$\begin{cases} u = x+y \\ v = x \end{cases} \Rightarrow \begin{cases} x = v \\ y = u-v \end{cases}$ $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$ (1 pt for correct $\frac{\partial(x,y)}{\partial(u,v)}$.)
For $0 < z < 10$, the corresponding region of D in the $uv-$plane is S which is bounded by $u+v=z \Rightarrow u=z$, $x=0 \Rightarrow v=0$, $y=0 \Rightarrow u=v$. (2 pts for correct corresponding region S.)



Hence

$$\iint_D \frac{1}{50}e^{-\frac{x}{5}}\,dA = \iint_S \frac{1}{50}e^{-\frac{v}{5}}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \int_0^z \int_0^u \frac{1}{50}e^{-\frac{v}{5}}\,dvdu$$

(1 pt for correct integrand $\frac{1}{50}e^{-\frac{v}{5}}$. 1 pt for correct upper bound of v, u. 1 pt for correct lower bound of v, 0.)

If students do not specify the corresponding region S and do change of variables directly, then they get 2 pts for the integrand, 2 pts for correct upper bound of v, and 1 pt for correct lower bound of v.

(c) (4%) Let $Z = X+Y$. Use the result from (b) to find the probability density function of Z, which is $\frac{d}{dz}P(X+Y \le z)$ for $0 < z < 10$.

Solution:

For $0 < z < 10$, the probability density function of $Z = X + Y$ is

$$\frac{d}{dz}\int_0^z \int_0^u \frac{1}{50}e^{-\frac{v}{5}}\,dvdu \qquad \text{(1 pt)}$$

$$= \int_0^z \frac{1}{50}e^{-\frac{v}{5}}\,dv \qquad \text{(2 pts)}$$

$$= \left(-\frac{1}{10}e^{-\frac{v}{5}}\right)\Bigg|_{v=0}^{v=z} = -\frac{1}{10}e^{-\frac{z}{5}} + \frac{1}{10} \qquad \text{(1 pt)}$$

8. (a) (4%) Write down the Taylor series of $\int_0^x e^{-t^2}\,dt$ at $x = 0$.

(b) (4%) Write down the Taylor series of $x\ln(1+3x)$ at $x = 0$.

(c) (3%) Compute $\lim_{x\to 0} \dfrac{\int_0^x e^{-t^2}\,dt - x}{x\ln(1+3x) - 3x^2}$.

Solution:

(a) $\int_0^x e^{-t^2}\,dt = \int_0^x \sum_{n=0}^{\infty} \frac{1}{n!}(-t^2)^n\,dt$

(1 pt for knowing the Maclaurin series of e^x, 1 pt for substituting $x = -t^2$ into the series of e^x.)

Thus $\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n}{n!} t^{2n}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{1}{2n+1} x^{2n+1}$.

(2 pts for correctly integrate the series term by term)

(b) $x\ln(1+3x) = x\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(3x)^n\right)$

(2 pts for knowing the Maclaurin series of $\ln(1+t)$. 1pt for substituting $t = 3x$ into the series of $\ln(1+t)$)

Hence $x\ln(1+3x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 3^n x^{n+1}$

(1 pt for final answer)

(c)

$$\lim_{x\to 0} \frac{\int_0^x e^{-t^2}\,dt - x}{x\ln(1+3x) - 3x^2} = \lim_{x\to 0} \frac{-\frac{1}{3}x^3 + \frac{1}{10}x^5 - \cdots}{-\frac{9}{2}x^3 + 9x^4 - \cdots} = \lim_{x\to 0} \frac{-\frac{1}{3} + \frac{1}{10}x^2 - \cdots}{-\frac{9}{2} + 9x - \cdots} = \frac{1}{3} \times \frac{2}{9} = \frac{2}{27}.$$

2 pts for first 1 or 2 nonzero terms of the series. 1 pt for the final answer.

If students derive wrong first non-zero terms of the series and compute the limit as the division of first non-zero coefficients, they get 1 pt.

If students try to compute the limit by L'Hospital rule and correctly compute the first derivatives of the denominator and numerator, they get 1 pt.