1102模組06-12班 微積分3 期考解答和評分標準

1. Suppose we are given a function $f(x,y)$ whose second order partial derivatives are continuous. Fix two points $P=(1,-1)$ and $Q=(1.2,-1.1)$ on the xy-plane. It is known that :

- $\langle -3,2,2\rangle$ is a normal vector to the surface $z=f(x,y)$ at $(1,-1,f(P))$,
- $f(x,y)$ attains an extreme value at Q.

Answer the following questions.

(a) (2%) Find $\dfrac{\partial f}{\partial x}(1,-1)$ and $\dfrac{\partial f}{\partial y}(1,-1)$.

> **Solution:**
> Since $(f_x(1,-1), f_y(1,-1), -1)$ is normal to the surface $z=f(x,y)$ at $(1,-1,f(P))$,
> we have $(f_x(1,-1), f_y(1,-1), -1)=\lambda(-3,2,2)$ for some λ (or $(f_x(1,-1), f_y(1,-1), -1) /\!\!/ (-3,2,2)$). (1%)
> So $\lambda=-1/2$, $f_x(1,-1)=3/2$ and $f_y(1,-1)=-1$. (1%)

(b) (4%) Use the linearization of $f(x,y)$ at $P=(1,-1)$ to estimate the value of $f(1.2,-1.1)-f(1,-1)$.

> **Solution:**
> The linearization of f at $(1,-1)$ is
>
> $$L(x,y)=f(1,-1)+f_x(1,-1)(x-1)+f_y(1,-1)(y+1)=f(1,-1)+\frac{3}{2}(x-1)-(y+1).(2\%)$$
>
> Then
>
> $$f(1.2,-1.1)-f(1,-1)\approx L(1.2,-1.1)-f(1,-1)=\frac{3}{2}(1.2-1)-(-1.1+1)=0.4.(2\%)$$

(c) (1%) (Circle the best answer.) $f(Q)$ is a
(i) maximum value (ii) minimum value

> **Solution:**
> Answer is (i). (1%)
> Since $f(Q)$ is an extreme value and $f(Q)>f(P)$ by (b), we have $f(Q)$ is a maximum value.

(d) (1%) (Circle the best answer.) If $f_{xy}(Q)\neq 0$, then $f_{xx}(Q)$ is
(i) positive (ii) non-negative (iii) zero (iv) non-positive (v) negative

> **Solution:**
> Answer is (v). (1%)
> Since $f(Q)$ is a maximum value, we have $f_{xx}(Q)\leq 0$. If $f_{xx}(Q)=0$, we have $D(Q)=-[f_{xy}(Q)]^2<0$ which implies that Q is a saddle point of f. It contradicts to (c) and we have $f_{xx}(Q)$ is negative.

2. Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

(a) (4%) Show that $f(x,y)$ is continuous at $(0,0)$.

(b) (6%) Find $\lim_{(x,y)\to(0,0)} f_x(x,y)$. Is $f_x(x,y)$ continuous at $(0,0)$? Explain.

(c) (5%) Let $\mathbf{u} = \langle a, b\rangle$ be a unit vector. Use the definition of directional derivatives to find $D_{\mathbf{u}}f(0,0)$. (Express your answer in terms of a and b.)

(d) (3%) Using (c), explain why $f(x,y)$ is not differentiable at $(0,0)$.

Solution:

(a) **Sol 1:**
$|f(x,y)| = \left|\frac{y^2}{x^2+y^2}x\right| = \frac{y^2}{x^2+y^2}|x| \le |x| \le \sqrt{x^2+y^2}$.
Hence $|f(x,y)| \to 0$ as (x,y) approaches (0,0). (3 pts for showing that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$)
Since $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, we conclude that f is continuous at (0,0).
(1 pt for showing that $f(x,y)$ is continuous at (0,0).)

Sol 2:
By polar coordinates,

$$|f(r\cos\theta, r\sin\theta)| = \left|\frac{r^3\cos\theta\sin^2\theta}{r^2}\right| = |r||\cos\theta|\sin^2\theta \le |r| \to 0 \text{ as } r \to 0.$$

Hence $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r\cos\theta, r\sin\theta) = 0$ (3 pts for showing that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$)
Since $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, we conclude that f is continuous at (0,0).
(1 pt for showing that $f(x,y)$ is continuous at (0,0).)

(b) For $(x,y) \neq (0,0)$, $f_x = \frac{y^2(x^2+y^2) - xy^2\cdot 2x}{(x^2+y^2)^2} = \frac{y^4 - x^2y^2}{(x^2+y^2)^2}$ (2 pts).
$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0.0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$ (1 pt)
$f_x(0,y) = \frac{y^4}{y^4} = 1$ for all $y \neq 0$.
Hence $f_x(x,y) \to 1 \neq f_x(0,0)$ as (x,y) approaches (0,0) along the y-axis.
This shows that $f_x(x,y)$ is not continuous at (0,0).
(Or we can also show that $f_x(x,y) \to 0$ as (x,y) approaches (0,0) along the x-axis. Hence $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist. $f_x(x,y)$ is not continuous at (0,0).)
(3 pts for showing that $f_x(x,y)$ is not continuous at (0,0).)

(c)

$$D_{\mathbf{u}}f(0,0) = \lim_{h\to 0}\frac{f(ah,bh) - f(0,0)}{h} \qquad \text{(2 pts for definition of } D_{\mathbf{u}}f(0,0))$$

$$= \lim_{h\to 0}\frac{\frac{h^3ab^2}{h^2} - 0}{h} = \lim_{h\to 0} ab^2 = ab^2 \qquad \text{(3 pts for the answer)}$$

(d) If $f(x,y)$ is differentiable at (0,0), the $D_{\mathbf{u}}f(0,0) = af_x(0,0) + bf_y(0,0)$. (2 pts)
However $D_{\mathbf{u}}f(0,0) = ab^2 \neq f_x(0,0)a + f_y(0,0)b$.
Therefore we know that f us not differentiable at (0,0). (1 pt)

3. Consider the function $I(x, y) = \int_{1-y}^{x} (t^2 + 3t) \cdot e^{t^2}\, dt$.

(a) (6%) Find all the critical points of $I(x, y)$.

(b) (6%) Classify the critical points of $I(x, y)$ as local maxima, local minima or saddle points.

Solution:

(a) The partial derivatives are

$$I_x = (x^2 + 3x)e^{x^2}, \text{(2 points)}$$

$$I_y = [(1-y)^2 + 3(1-y)]e^{(1-y)^2}. \text{(2 points)}$$

The critical points will be the solutions of $I_x = 0$ and $I_y = 0$. Namely, $x^2 + 3x = 0$ and $(1-y)^2 + 3(1-y) = 0$ which yield $x = 0, -3$ and $y = 1, 4$. So the critical points are $(0, 1), (0, 4), (-3, 1)$, and $(-3, 4)$.(2 points)

(b) The critical points we found in (a) all have gradient zero, so we use Second Derivatives Test to classify them. We compute I_{xx}, I_{xy}, and I_{yy}.

$$I_{xx} = (2x + 3)e^{x^2} + (x^2 + 3x)e^{x^2} 2x,$$

$$I_{yy} = [2(y-1) - 3]e^{(1-y)^2} + [(1-y)^2 + 3(1-y)]e^{(1-y)^2} 2(y-1).$$

Both I_{xy} and I_{yx} are zero. So

$$D = I_{xx}I_{yy} - I_{xy}^2 = [2x + 3 + (x^2 + 3x)2x]e^{x^2} e^{(1-y)^2} \big(2y - 5 + [(1-y)^2 + 3(1-y)]2(y-1)\big). \text{(2 points)}$$

$D(0, 1) = -9 < 0$, so a saddle point. $D(0, 4) = 9e^9 > 0$, and $I_{xx}(0, 4) = 3 > 0$, so a local min. $D(-3, 1) = 9e^9 > 0$, and $I_{xx}(-3, 1) = -3e^9 < 0$, so a local max. Finally, $D(-3, 4) = -9e^{18} < 0$, a saddle point. In summary, $(0, 1)$ and $(-3, 4)$ are saddle points, $(0, 4)$ is a local min, and $(-3, 1)$ is a local max. (1 point for each critical point.)

4. (12%) By the method of Lagrange multipliers, find the absolute maximum and minimum values of

$$f(x,y,z) = x^2 - 2y^2 - 2z^2 + 4xz$$

on the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution:

Let

$$f(x,y,z) = x^2 - 2y^2 - 2z^2 + 4xz,$$
$$g(x,y,z) = x^2 + y^2 + z^2 - 1.$$

By the method of Lagrange multiplier, we have the set of equations

$$2x + 4z = 2\lambda x \quad (1)$$
$$-4y = 2\lambda y \quad (2)$$
$$4x - 4z = 2\lambda z \quad (3)$$

along with $g(x,y,z) = 0$. Equation (2) gives either $y = 0$ or $\lambda = -2$. If $\lambda = -2$, the system becomes a set of linear equations

$$3x + 2z = 0$$
$$x = 0$$

from which we get $(x,y,z) = \pm(0,1,0)$. If $y = 0$, we can eliminate λ by $x \times (3) - z \times (1)$ and obtain

$$2x^2 - 3xz - 2z^2 = (2x+z)(x-2z) = 0,$$

hence $z = \frac{x}{2}, -2x$. If $z = \frac{x}{2}$, we have $g(x,0,\frac{x}{2}) = \frac{5}{4}x^2 - 1 = 0$, from which we get $(x,y,z) = \pm\left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$. If $z = -2x$, we have $g(x,0,-2x) = 5x^2 - 1 = 0$, from which we get $(x,y,z) = \pm\left(\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}\right)$. The values of f at those six critical points are

$$f(\pm(0,1,0)) = -2,\ f(\pm(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}})) = 2,\ f(\pm(\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}})) = -3,$$

respectively, so the maximum is 2 and the minimum is -3.

Marking scheme

- Setting up the equation of the Lagrange multiplier correctly (4%).
  - Incorrect equations with a very minor mistakes (an incorrect sign etc.) (3%).
- 1% for finding each of the six solutions correctly (6% in total).
  - 1% for being aware of the existence of the two cases $\lambda = -2$ and $y = 0$ (with incomplete calculation)
  - 1% for finiding the fact $x = 2z, -z/2$ (or the factorization $(x-2z)(2x+z) = 0$) for $y = 0$ case.
- Finding correct maximal value (1%), minimal value (1%)

5. Depicted in the **Figure**, E is an 'apple-shaped' solid that, in *spherical coordinates*, occupies the region

$$E = \{(\rho, \phi, \theta) : 0 \le \rho \le 1 - \cos\phi,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}.$$

It is known that E has a constant density $\rho(x, y, z) = 2$.

(a) (6%) Find the mass of E.

(b) (8%) Let $(0, 0, \overline{z})$ be the center of mass of E. Find $\overline{z}$.

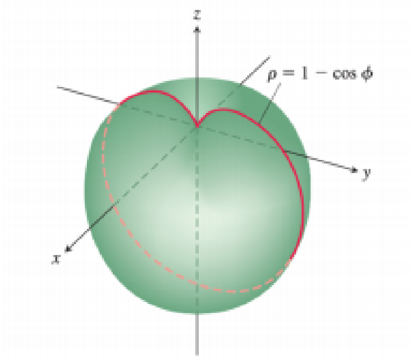

Figure. The apple-shaped solid E

Solution:

Prob.5: The following crucial steps must be shown clearly

(a) Total mass:

$M = \int\int\int_E 2\,dV = 2\int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta$ 2%

$= 4\pi \int_0^{\pi} \frac{(1-\cos\phi)^3}{3} \sin\phi\, d\phi$ 2%

either $= \frac{4\pi}{3}\int_{-1}^{1}(1-x)^3 dx = \frac{4\pi}{3}[\frac{(1-x)^4}{4}]_1^{-1} = \frac{16\pi}{3}$

or $= \frac{4\pi}{3}[\frac{(1-\cos\phi)^4}{4}]_0^{\pi} = \frac{16\pi}{3}$ 2%

(b) The z-component of center of mass:

$\bar{z} = \frac{1}{M}\int\int\int_E 2z\,dV = \frac{2}{M}\int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} \rho^3 \sin\phi\cos\phi\, d\rho d\phi d\theta$ 3%

$= \frac{\pi}{M}\int_0^{\pi}(1-\cos\phi)^4 \sin\phi\, \cos\phi\, d\phi$ 2%

either $= \frac{\pi}{M}\int_{-1}^{1} x(1-x)^4 dx = \frac{3}{16}[\frac{(1-x)^5}{5} - \frac{(1-x)^6}{6}]_1^{-1} = \frac{-4}{5}$

or $= \frac{\pi}{M}[\frac{(1-\cos\phi)^5}{5} - \frac{(1-\cos\phi)^6}{6}]_0^{\pi} = \frac{3}{16}\frac{-128}{30} = \frac{-4}{5}$ 3%

6. (a) (9%) Find the volume of the solid that is below the parabolid $z = 9 - x^2 - y^2$ and above the region enclosed by the lemniscate $r^2 = \cos(2\theta)$ on the xy-plane (See **Figure**).

(b) (9%) Evaluate the triple integral

$$\iiint_R (36 - x^2 - 4y^2 - 9z^2)\ \mathrm{d}V$$

where $R = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 4y^2 + 9z^2 \leq 36\}$.

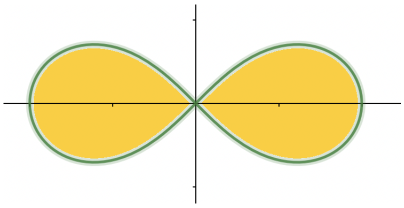

Figure. The lemniscate $r^2 = \cos(2\theta)$

Solution:

(a) In cylindrical coordinates, the volume is

$$\int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos(2\theta)}} \int_0^{9-r^2} r\ dz\ dr\ d\theta + \int_{3\pi/4}^{5\pi/4} \int_0^{\sqrt{\cos(2\theta)}} \int_0^{9-r^2} r\ dz\ dr\ d\theta$$

By symmetry, the two integrals are the same value.

$$= 2 \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos(2\theta)}} \int_0^{9-r^2} r\ dz\ dr\ d\theta = 2 \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos(2\theta)}} 9r - r^3\ dr\ d\theta$$

$$= 2 \int_{-\pi/4}^{\pi/4} \frac{9\cos(2\theta)}{2} - \frac{\cos^2(2\theta)}{4}\ d\theta = \int_{-\pi/4}^{\pi/4} 9\cos(2\theta) - \frac{1 + \cos(4\theta)}{4}\ d\theta = 9 - \frac{\pi}{8}$$

(b) Let $x = 6u,\ y = 3v,\ z = 2w$, then the triple integral becomes

$$\iiint_{u^2+v^2+w^2\leq 1} (36 - 36x^2 - 36y^2 - 36z^2)|J|\ dV = 6^4 \iiint_{u^2+v^2+w^2\leq 1} (1 - u^2 - v^2 - w^2)\ dV$$

Use spherical coordinates $u = \rho\sin\phi\cos\theta,\ v = \rho\sin\phi\sin\theta,\ w = \rho\cos\phi$.

$$= 6^4 \int_0^{2\pi} \int_0^{\pi} \int_0^1 (1 - \rho^2)\rho^2 \sin\phi\ d\rho\ d\phi\ d\theta = (6^4)(2\pi)(2)\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{3456\pi}{5}$$

Grading: In general, -2% for each big mistake and -1% for each small mistake.

(a) 5% for cylindrical coordinates integral setup, 4% for computation (they get partial credit in computation if their setup is similar to the answer).

(b) 5% for setting up an integral that they can evaluate, 4% for computation (they get partial credit in computation if their setup is similar to the answer).

Note: Both problems can be evaluated using xyz-coordinates.

7. (a) (i) (2%) For each fixed value of y, find $\int_1^{\sqrt{3}} \cos(xy)\,\mathrm{d}x$.

(ii) (6%) Use your result in (i) to transform

$$I = \int_0^\infty \frac{e^{-y}\cdot\left(\sin(\sqrt{3}y) - \sin(y)\right)}{y}\,\mathrm{d}y$$

into a double integral and then evaluate I by Fubini's Theorem.

(b) (10%) Use the change of variables $u = xy$ and $v = y$ to evaluate

$$\iint_R \frac{y}{1+x^2y^2}\,\mathrm{d}A$$

where R is the region enclosed by the curves $xy = 1$, $xy = \sqrt{3}$, $x = 1$ and $y = 3$.

Solution:

(a) (i) **Marking scheme for 7(a)(i).**
All or nothing. 1M for sign error.

$$\int_1^{\sqrt{3}} \cos(xy)\,dx = \left[\frac{\sin(xy)}{y}\right]_1^{\sqrt{3}} = \frac{\sin(\sqrt{3}y) - \sin(y)}{y}$$

(ii) **Marking scheme for 7(a)(ii).**

- (1M) Use (a) to transform I into a double integral
- (1M) Use Fubini's theorem (correctly)
- (2M)* Correct antiderivative for dy
- (1M) For realising $\lim_{y\to\infty} e^{-y}\cdot(\text{bounded function}) = 0$
- (1M) Correct answer

Remark for *. 1M for any candidates who apply IBP twice but yield incorrect anti-derivative.

$$\int_0^\infty \frac{e^{-y}\cdot\left(\sin(\sqrt{3}y) - \sin(y)\right)}{y}\,\mathrm{d}y \overset{(a)}{=} \underbrace{\int_0^\infty \int_1^{\sqrt{3}} e^{-y}\cdot\cos(xy)\,dx\,dy}_{(1M)}$$

$$\overset{\text{Fubini}}{=} \underbrace{\int_1^{\sqrt{3}} \int_0^\infty e^{-y}\cdot\cos(xy)\,dy\,dx}_{(1M)}$$

$$\overset{\text{IBP}}{=} \int_1^{\sqrt{3}} \left[\underbrace{\frac{\mathrm{e}^{-y}\cdot(x\sin(xy) - \cos(xy))}{x^2+1}}_{(2M)}\right]_0^\infty dx$$

$$= \int_1^{\sqrt{3}} \underbrace{\frac{1}{1+x^2}}_{(1M)}\,dx$$

$$= \underbrace{\frac{\pi}{12}}_{(1M)}$$

(b) **Marking scheme for 7(b).**

- (2M) correct Jacobian
- (1M+1M) correct u- and v-components of the transformed region (okay to just sketch the region)
- (3M) correct change of variable formula : integrand, integration limits for dv and du
- (1M+1M) correct antiderivatives
- (1M) correct answer

Remark. the first 7M will be awarded as long as a candidate transforms the integral perfectly.

Let $u = xy$ and $v = y$. Then we have $\begin{cases} x = \frac{u}{v} \\ y = v \end{cases}$ and the Jacobian equals to $\begin{vmatrix} \frac{1}{v} & \star \\ 0 & 1 \end{vmatrix} = \underbrace{\frac{1}{v}}_{(2M)}$. Moreover, the given region is transformed as a trapezoidal region

$$\{(u,v) : \underbrace{1 \le u \le \sqrt{3}}_{(1M)} \text{ and } \underbrace{u \le v \le 3}_{(1M)}\}.$$

$$\begin{aligned}
\iint_R \frac{y}{1+x^2y^2}\,\mathrm{d}A &= \underbrace{\int_1^{\sqrt{3}} \int_u^3 \frac{v}{1+u^2} \cdot \frac{1}{v}\,dv\,du}_{(3M)} \\
&= \int_1^{\sqrt{3}} \frac{3-u}{1+u^2}\,du \\
&= \left[3\underbrace{\tan^{-1}(u)}_{(1M)} - \underbrace{\frac{1}{2}\ln(1+u^2)}_{(1M)}\right]_1^{\sqrt{3}} \\
&= \underbrace{\frac{\pi}{4} - \frac{\ln 2}{2}}_{(1M)}
\end{aligned}$$