1. (20 pts) Let $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) (5 pts) Is $f(x,y)$ continuous at $(0,0)$? Explain.

(b) (5 pts) Use the definition of directional derivatives to find $D_{\langle a,b\rangle} f(0,0)$ for any unit vector $\langle a,b\rangle$.

(c) (5 pts) The function f is differentiable at $(0,0)$ if

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = 0$$

where $L(x,y)$ is the linearization of $f(x,y)$ at $(0,0)$. Is $f(x,y)$ differentiable at $(0,0)$? Explain.

(d) (5 pts) Find $f_y(x,y)$when $(x,y) \neq (0,0)$. Is $f_y(x,y)$ continuous at $(0,0)$? Explain.

Solution:

(a) We use the inequality

$$0 \le y^2 \le x^4 + y^2 \quad \Rightarrow 0 \le \frac{y^2}{x^4+y^2} \le 1 \text{ for any } (x,y) \neq (0,0).$$

Then for any $(x,y) \neq (0,0)$, we have

$$0 \le f(x,y) \le x^2, \quad \lim_{(x,y)\to(0,0)} 0 = 0, \quad \lim_{(x,y)\to(0,0)} x^2 = 0.$$

By squeeze theorem, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ and f is continuous at $(0,0)$.

(b)

$$D_{\langle a,b\rangle} f(0,0) = \lim_{t\to0} \frac{f(at,bt) - f(0,0)}{t} = \lim_{t\to0} \frac{a^2b^2t^4}{a^4t^5 + b^2t^3} = \lim_{t\to0} \frac{a^2b^2t}{a^4t^2+b^2}$$

Case 1: $b = 0$, $\langle a,b\rangle = \langle 1,0\rangle$,

$$D_{\langle 1,0\rangle} f(0,0) = f_x(0,0) = \lim_{t\to0} \frac{f(at,bt) - f(0,0)}{t} = 0.$$

Case 2: $b \neq 0$

$$D_{\langle a,b\rangle} f(0,0) = \lim_{t\to0} \frac{a^2b^2t}{a^4t^2+b^2} = 0.$$

(c) From (b), we know that $f_x(0,0) = 0$ and $f_y(0,0) = 0$.
Linearization $L(x,y) = 0 + 0x + 0y = 0$.
Consider $\lim_{(x,y)\to(0,0)} \dfrac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{(x^4+y^2)\sqrt{x^2+y^2}}$
Use squeeze theorem again with inequalities: $-1 \le \dfrac{x}{\sqrt{x^2+y^2}} \le 1$, $0 \le \dfrac{y^2}{x^4+y^2} \le 1$. Hence

$$-x \le \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} \le x, \quad \Rightarrow \quad \lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = 0$$

(d) From (b), $f_y(0,0) = 0$. For $(x,y) \neq (0,0)$,

$$f_y(x,y) = \frac{2x^2y(x^4+y^2) - 2x^2y^3}{(x^4+y^2)^2}$$

Along the path $\mathbf{r}(t) = \langle t, t^2\rangle$, $\lim_{t\to0} f(\mathbf{r}(t)) = \lim_{t\to0} \dfrac{4t^8 - 2t^8}{4t^8} \neq 0$.
So f_y is not continuous at $(0,0)$.

Grading:

(a) If students use polar but did not discuss $\sin\theta = 0$ case, (-3 pts). If students only show a few paths, (-4 pts). Otherwise, (-2 pts) for each logic/concept mistake.

(b) The definition is (3 pts). (2 pts) for evaluating the limit carefully. (-1 pt) if they did not consider $b = 0$.

(c) Linearization is worth (2 pts). Similar to (a), (-2 pts) if students use polar or paths incorrectly.

(d) (2 pts) for $f_y(0,0)$ and $f_y(x,y)$. (3 pts) for showing not continuous.

Note: Students can use change of variables ($y = z^2$) before switching to using polar coordinates, then the arguments are easier for each limit.

2. (9 pts) Find the equation of the tangent plane to the surface

$$r^2 - (1 - z^2)\cos(2\theta) = 0$$

where r, θ, and z are the cylindrical coordinates, at the point where $r = \frac{1}{\sqrt{3}}, \theta = \frac{\pi}{6}, z = \frac{1}{\sqrt{3}}$.

Solution:

In terms of the Cartesian coordinates, i.e.

$$(x, y, z) = (r\cos\theta, r\sin\theta, z),$$

the given point is

$$\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \qquad \text{(1 pt)}$$

and the equation of the surface is

$$\left(x^2 + y^2\right)^2 - \left(1 - z^2\right)\left(x^2 - y^2\right) = 0. \qquad \text{(3 pts)}$$

Let $f(x, y, z) = \left(x^2 + y^2\right)^2 - \left(1 - z^2\right)\left(x^2 - y^2\right)$. Then

$$\nabla f\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}\right) =$$

$$= \left(4x\left(x^2 + y^2\right) - 2x\left(1 - z^2\right), 4y\left(x^2 + y^2\right) + 2y\left(1 - z^2\right), 2z\left(x^2 - y^2\right)\right)\Big|_{\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

$$= \left(0, \frac{4}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}\right). \qquad \text{(3 pts)}$$

Thus, the equation of the tangent plane to $f(x, y, z) = 0$ at $\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is

$$\left(0, \frac{4}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}\right) \cdot \left(x - \frac{1}{2}, y - \frac{1}{2\sqrt{3}}, z - \frac{1}{\sqrt{3}}\right) = 0. \qquad \text{(2 pts)}$$

3. (9 pts) Suppose that the temperature at the point $(x, y, z) \in \mathbb{R}^3$ is indicated by a C^1 function $f(x, y, z)$. One launches three observers at time $t = 0$, which travels along the curves

$$\begin{aligned}\gamma_1(t) &= (1 + t^2, 2 + 4t^3, 3 + \sin t),\\ \gamma_2(t) &= (1 + t^4, 1 + e^t, 2 + e^{2t}), \quad \text{and}\\ \gamma_3(t) &= (e^t, 1 + e^{3t}, 3 + 4t), \quad \text{respectively.}\end{aligned}$$

Suppose that the observe temperature data are given by

$$f(\gamma_1(t)) = 6 + 5t + 3t^2, \; f(\gamma_2(t)) = 5 + 3t + e^{4t}, \quad \text{and} \quad f(\gamma_3(t)) = 6 + 3\sin(5t), \text{ respectively.}$$

(a) (6 pts) Find $\nabla f(1, 2, 3)$.

(b) (3 pts) Use linear approximation to estimate $f(1.04, 2.01, 3.02)$.

Solution:

Note that $\gamma_1(0) = (1, 2, 3)$ and hence

$$f(1, 2, 3) = f(\gamma_1(0)) = 6.$$

We have

$$\gamma_1'(0) = (0, 0, 1), \; \gamma_2'(0) = (0, 1, 2), \quad \text{and} \quad \gamma_3'(0) = (1, 3, 4).$$

Denote $\nabla f(1, 2, 3)$ by (A, B, C). Since $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = (1, 2, 3)$, we have by the chain rule that

$$(f \circ \gamma_i)'(0) = \nabla f(1, 2, 3) \cdot \gamma_i'(0) \; (i = 1, 2, 3),$$

and hence

$$\begin{cases} 0A + 0B + 1C = 5 \\ 0A + 1B + 2C = 7 \\ 1A + 3B + 4C = 15. \end{cases}$$

This implies that $\nabla f(1, 2, 3) = (A, B, C) = (4, -3, 5)$. The linear approximation of $f(x, y, z)$ at $(1, 2, 3)$ is then

$$f(1, 2, 3) + \nabla f(1, 2, 3) \cdot (x - 1, y - 2, z - 3) = 6 + 4(x - 1) - 3(y - 2) + 5(z - 3).$$

Therefore, the estimated value of $f(1.04, 2.01, 3, 02)$ is

$$6 + 4(1.04 - 1) - 3(2.01 - 2) + 5(3.02 - 3) = 6.23.$$

Grading scheme. *(a)* 的分數分為以下四部分:

(a1) *(3分)* 提及方程組$(f \circ \gamma_i)'(0) = \nabla f(1, 2, 3) \cdot \gamma_i'(0) \; (i = 1, 2, 3)$或是其等價物。以下情況可得部分分數:

- (*1*分) 僅提到 *chain rule* 但未列出方程組$(f \circ \gamma_i)'(0) = \nabla f(1, 2, 3) \cdot \gamma_i'(0)$。

(a2) *(1分)* 正確算出$\gamma_i'(0) \; (i = 1, 2, 3)$三數。

(a3) *(1分)* 正確算出$(f \circ \gamma_i)'(0) \; (i = 1, 2, 3)$三數。

(a4) *(1分)* 利用*(a2)*與*(a3)*所得的 (可能是錯誤的) 資訊,正確地解了方程組*(a1)*的方程組。

(b) 的分數分為以下兩部分:

(b1) *(1分)* 正確求得$f(1, 2, 3)$。

(b2) *(1分)* 利用*(a)*所求得的結果正確寫出線性逼近的形式。

(b3) *(1分)* 正確求出線性逼近的數值。

4. (10 pts) Let $f(x,y) = 2x^3 + 2xy^2 - 3x^2 + y^2$.

(a) (4 pts) Find all critical points of f.

(b) (4 pts) Classify each of the critical points by the second partial derivatives test.

(c) (2 pts) Does f attain an absolute minimum? Does f attain an absolute maximum? Explain your answers.

Solution:

(a) We have
$$f_x(x,y) = 6x^2 + 2y^2 - 6x \quad \text{and} \quad f_y(x,y) = 4xy + 2y.$$

We solve (x,y) for the equation $\nabla f(x,y) = (0,0)$:

$$\begin{cases} 6x^2 + 2y^2 - 6x = 0 \\ 4xy + 2y = 0 \end{cases} \Leftrightarrow \begin{cases} 3x(x-1) + y^2 = 0 \\ (2x+1)y = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} 3x(x-1) + y^2 = 0 \\ x = -1/2 \text{ or } y = 0 \end{cases} \Leftrightarrow (x,y) = (0,0) \text{ or } (1,0).$$

The critical points of f are $(0,0)$ and $(1,0)$.

(b) We have
$$\begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix} = \begin{pmatrix} 12x-6 & 4y \\ 4y & 4x+2 \end{pmatrix},$$

and hence
$$D(0,0) = \det\begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} = -12 < 0 \quad \text{and} \quad D(1,0) = \det\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = 36 > 0.$$

Therefore, $(0,0)$ is a saddle point of f; since $f_{xx}(1,0) = 6 > 0$, $(1,0)$ is a local minimum point of f.

(c) f has neither absolute minimum nor maximum since $f(x,0) = 2x^3 - 3x^2$ is neither bounded from below nor from above.

Grading scheme. *(a)* 的分數分為以下兩部分:

(a1) (2分) 正確算出$\nabla f(x,y)$並列出方程組$\nabla f(x,y) = (0,0)$。不給部分分數。

(a2) (2分) 解$\nabla f(x,y) = (0,0)$得 *critaical points* $(0,0)$與$(1,0)$。以下情況可得部分分數:

- *(1分)* $\nabla f(x,y) = (0,0)$列式錯誤，但解方程式過程正確，且解得兩個以上的 *critical points*（未必為$(0,0)$或$(1,0)$）。
- *(1分)* $\nabla f(x,y) = (0,0)$列式正確，但解方程式過程有誤，且解得兩個以上的 *critical points*（未必為$(0,0)$或$(1,0)$）。

(b) 的分數分為以下兩部分:

(b1) (2分) 正確算出$D(x,y)$在所有*(a)*部分所解得的 *critical points* 處的值。以下情況可得部分分:

- （*1*分）有正確算出函數$D(x,y)$，但是代入所*(a)*解得的 *critical points* 時有任何計算錯誤。

(b2) (2分) 假設*(a)*中所解得的 *critical points* 至少有兩個點。在每個點處正確運用二階偏導數檢驗法搭配*(b1)*所求得的D與f_{xx}（或f_{yy}）的符號判斷局部極值的種類。以下情況可得部分分數:

- （*1*分）*(a)* 中解得的 *critical point* 至少有兩個，但是使用二階判別法的方式僅部分正確（不論D與f_{xx}或f_{yy}的值是否正確）。
- （*1*分）*(a)* 中解得的 *critical point* 只有個，但是使用二階判別法的方式完全正確（不論D與f_{xx}或f_{yy}的值是否正確）。

(c)（*2*分）部分絕對最大值、絕對最小值不存在的理由各佔*1*分，均無部分分數。

5. (12 pts) Let $f(x,y) = 2x^2 - 2xy + 3y^2$. For $(a,b) \in \mathbb{R}^2$, we define $h(a,b)$ to be the maximum rate of increase of f at (a,b).

(a) (4 pts) Find $h(4,4)$ and the direction it occurs in.

(b) (8 pts) Use the Lagrange multipliers method to find the absolute maximum and absolute minimum values of h on the constraint $a^2 - ab + b^2 = 16$.

Solution:

(a) The gradient of f at $(4,4)$ is

$$\nabla f(4,4) = (4x - 2y, -2x + 6y)|_{(4,4)} = (8,16). \qquad \text{(2 pts)}$$

So

$$h(4,4) = |(8,16)| = 8\sqrt{5} \qquad \text{(1 pt)}$$

and the maximum rate is attained in the direction

$$\frac{(8,16)}{|(8,16)|} = \frac{1}{\sqrt{5}}(1,2). \qquad \text{(1 pt)}$$

(b) Finding the extrema of

$$h(x,y) = |\nabla f(x,y)| = 2\sqrt{5}\left(x^2 - 2xy + 2y^2\right)^{\frac{1}{2}}.$$

subject to the constraint $x^2 - xy + y^2 = 16$ is equivalent to finding the extrema of

$$x^2 - 2xy + 2y^2$$

subject to the same constraint. Note that both of the maxima and minima exist by the Extreme Value Theorem (since the ellipse $x^2 - xy + y^2 = 16$ is a bounded closed set) and they satisfy

$$\nabla\left(x^2 - 2xy + 2y^2\right) = \lambda \nabla\left(x^2 - xy + y^2\right), \qquad \text{(1 pt)}$$

i.e.

$$(2x - 2y, -2x + 4y) = \lambda(2x - y, -x + 2y). \qquad \text{(2 pts)}$$

In view of the second components of both sides, one can infer that either $x = 2y$ or $\lambda = 2$; in the latter case one has $x = 0$. Substituting into the constraint $x^2 - xy + y^2 = 16$ yields

$$(x,y) \in \left\{\pm\left(\frac{8}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right), \pm(0,4)\right\}. \qquad \text{(3 pts)}$$

It follows that

$$h\left(\frac{8}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = h\left(-\frac{8}{\sqrt{3}}, -\frac{4}{\sqrt{3}}\right) = 8\sqrt{\frac{10}{3}} \qquad \text{(1 pt)}$$

is the minimum value and

$$h(0,4) = h(0,-4) = 8\sqrt{10} \qquad \text{(1 pt)}$$

is the maximum value.

6. (10 pts) Fill in the blanks. Do not evaluate.

(a) (6 pts) Let E be the solid below the paraboloid $S_1 : z = 5 - x^2 - y^2$ and above the hemisphere $S_2 : z = \sqrt{5 - x^2 - y^2}$.

The volume of $E = \int_{\_\_\_}^{\_\_\_} \int_{\_\_\_}^{\_\_\_} \_\_\_\_\_\_\_\_ \, dr \, d\theta$.

The surface area of $E =$ The area on $S_1 +$ The area on S_2

$$= \int_{\_\_\_}^{\_\_\_} \int_{\_\_\_}^{\_\_\_} \_\_\_\_\_\_\_\_ \, dr \, d\theta + \int_{\_\_\_}^{\_\_\_} \int_{\_\_\_}^{\_\_\_} \_\_\_\_\_\_\_\_ \, dr \, d\theta$$

(b) (4 pts) Let D be the region described by

$$\begin{cases} x + y \le 3, \\ x + z^2 \le 4, \\ x \ge 0, y \ge 0, z \ge 0. \end{cases}$$

Then

$$\iiint_D F(x,y,z) \, dV = \int_{\_\_\_}^{\_\_\_} \int_{\_\_\_}^{\_\_\_} \int_{\_\_\_}^{\_\_\_} F(x,y,z) \, dz \, dy \, dx.$$

Solution:

(a) The volume of $E = \int_0^{2\pi} \int_0^2 r(5 - r^2 - \sqrt{5 - r^2}) \, dr \, d\theta$.

The surface area of $E =$ The area on $S_1 +$ The area on S_2

$$= \int_0^{2\pi} \int_0^2 r\sqrt{1 + 4r^2} \, dr \, d\theta + \int_0^{2\pi} \int_0^2 r\sqrt{\frac{5}{5 - r^2}} \, dr \, d\theta$$

(b)

$$\iiint_D F(x,y,z) \, dV = \int_0^3 \int_0^{3-x} \int_0^{\sqrt{4-x}} F(x,y,z) \, dz \, dy \, dx.$$

Grading:

(a) Integration bounds are the same for all three (2 pts), Jacobian for polar (1 pt), and (1 pt) for each integrand function.

(b) (-1 pt) for each mistake.

7. (12 pts) Compute the following integrals.

(a) (6 pts) $\iint_R xe^{2y^3-3y^2}\,dA$, where R is bounded by the curves $y = x$ and $y = x^2$.

(b) (6 pts) $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1+\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$.

Solution:

(a) R is a region of both type I and type II. Since the integration w.r.t. y is almost impossible, we choose to first integrate w.r.t. x. Thus we write R as a type II region, $R = \{(x,y)|0 \le y \le 1, y \le x \le \sqrt{y}\}$

$$\iint_R xe^{2y^3-3y^2}\,dA = \int_0^1\int_y^{\sqrt{y}} xe^{2y^3-3y^2}dxdy \text{(3 pts for expressing the double integral as this iterated integral.)}$$

$$= \int_0^1 \frac{1}{2}(y-y^2)e^{2y^3-3y^2}\,dy \text{(1 pt for integrating w.r.t. } x\text{)}$$

$$= \left(-\frac{1}{12}e^{2y^3-3y^2}\right)\Big|_{y=0}^{y=1} = \frac{1}{12}(1-e^{-1}) \text{(2 pts for integrating w.r.t. } y \text{ and the final answer)}$$

If students write $\iint_R xe^{2y^3-3y^2}\,dA = \int_0^1\int_{x^2}^{x} xe^{2y^3-3y^2}dydx$, they get 2 pts out of 6 pts.

(b) **Solution 1:**

The iterated integral is $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$, where E is a type I solid with the upper boundary surface, $z = 1+\sqrt{1-x^2-y^2}$ i.e. $x^2+y^2+z^2 = 2z$, $z \ge 1$, and the lower boundary surface $z = \sqrt{x^2+y^2}$.
The projection of E onto the xy-plane is $D = \{(x,y)|x^2+y^2 \le 1, y \ge 0\}$, the upper half unit disk.
In Spherical coordinates, the upper boundary surface is $\rho = 2\cos\varphi$ and the lower boundary surface is $\varphi = \frac{\pi}{4}$.
Moreover, D tells us the range of θ is $[0,\pi]$.
Hence $R = \{(\rho,\varphi,\theta)|0 \le \theta \le \pi, 0 \le \varphi \le \frac{\pi}{4}, 0 \le \rho \le 2\cos\varphi\}$

$$\iiint_E \sqrt{x^2+y^2+z^2}\,dV = \int_0^{\pi}\int_0^{\frac{\pi}{4}}\int_0^{2\cos\varphi} \rho\cdot\rho^2\sin\varphi\,d\rho d\varphi d\theta$$

(1 pt for correct integrand, 3 pts for correct range of ρ, φ and θ)

$$= \int_0^{\pi}\int_0^{\frac{\pi}{4}} 4\cos^4\varphi\sin\varphi\,d\varphi d\theta \text{ (1 pt for integration w.r.t. } \rho\text{)}$$

$$\xlongequal[du=-\sin\varphi\ d\varphi]{u=\cos\varphi} 4\int_0^{\pi}\int_1^{\frac{1}{\sqrt{2}}} u^4(-du)d\theta = 4\pi\left(\frac{u^5}{5}\Big|_{u=\frac{1}{\sqrt{2}}}^{u=1}\right) = \frac{4}{5}\pi(1-\frac{1}{4\sqrt{2}})$$

(1 pt for integration e.r.t. φ and the final answer)

Solution 2:
In Cylindrical coordinates, $E = \{(r,\theta,z)|0 \le \theta \le \pi, 0 \le r \le 1, r \le z \le 1+\sqrt{1-r^2}\}$.

$$\iiint_E \sqrt{x^2+y^2+z^2}\,dV = \int_0^{\pi}\int_0^1\int_r^{1+\sqrt{1-r^2}} \sqrt{r^2+z^2}\cdot r\cdot dzdrd\theta$$

(1 pt for correct integrand, 2 pts for correct ranges of z, r, and θ.)

8. (18 pts) Compute the following integrals by making a suitable change of variables.

(a) (9 pts) $\iint_T e^{(x-2y)^2}\, dA$, where T is the triangle bounded by $x-2y=1, 3x+y=0$, and $x+5y=0$.

(b) (9 pts) $\iint_D (x^3y+5y^3x)\, dA$, where $D=\{(x,y)\mid 3\le x^2+y^2\le 4, 1\le x^2-y^2\le 2, x\ge 0, y\ge 0\}$.

Solution:

(a) **Solution 1:**

Let $\begin{cases} u=x-2y \\ v=3x+y \end{cases}$. Then $\begin{cases} x=\frac{1}{7}(u+2v) \\ y=\frac{1}{7}(v-3u) \end{cases}$. $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{1}{7} & \frac{2}{7} \\ -\frac{3}{7} & \frac{1}{7} \end{vmatrix}=\frac{1}{7}$

(1 pt for choosing new variables u and v. 2 pts for the Jacobian)

The corresponding region of T in the uv-plane is S, which is bounded by ① $x-2y=1 \Rightarrow u=1$, ② $3x+y=0\Rightarrow v=0$ ③ $x+5y=0\Rightarrow \frac{1}{7}[(u+2v)+5(v-3u)]=0\Rightarrow -14u+7v=0\Rightarrow v=2u$

Hence $\iint_T e^{(x-2y)^2}\, dA=\iint_S e^{u^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$

(1 pt for the integrand $e^{u^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$. 2 pts for correct region S.)

$$\iint_S e^{u^2}\cdot\frac{1}{7}\, dudv=\int_0^1\int_0^{2u}\frac{1}{7}e^{u^2}dvdu \quad \text{(1 pt for correct iterated integral)}$$
$$=\frac{1}{7}\int_0^1 2ue^{u^2}du=\frac{1}{7}(e^{u^2})\Big|_{u=0}^{u=1}=\frac{1}{7}(e-1)$$

(1 pt for integration w.r.t. u. 1 pt for the final answer.)

Solution 2:

Let $\begin{cases} u=x-2y \\ v=x+5y \end{cases}$. Then $\begin{cases} x=\frac{5u+2v}{7} \\ y=\frac{v-u}{7} \end{cases}$, $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{5}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} \end{vmatrix}=\frac{1}{7}$

(1 pt for choosing new variables u and v. 2 pts for $\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{7}$)

The corresponding region of T in the uv-plane is S which is bounded by $u=1$, $v=0$ and $3x+y=0\Rightarrow \frac{1}{7}[3(5u+2v)+v-u]=0\Rightarrow 14u+7v=0\Rightarrow v=-2u$

Hence $\iint_T e^{(x-2y)^2} dA=\iint_S e^{u^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$

(1 pt for the integrand $e^{u^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$. 2 pts for correct region S.)

$$\iint_S e^{u^2}\frac{1}{7}\, dudv=\frac{1}{7}\int_0^1\int_{-2u}^0 e^{u^2}\, dvdu \quad \text{(1 pt for correct iterated integral.)}$$
$$=\frac{1}{7}\int_0^1 2ue^{u^2}\, du=\frac{1}{7}(e-1) \quad \text{(2 pts for integration)}$$

(b) Let $\begin{cases} u=x^2+y^2 \\ v=x^2-y^2 \end{cases}$. For $x\ge 0, y\ge 0$, we have $\begin{cases} x=\sqrt{\frac{u+v}{2}} \\ y=\sqrt{\frac{u-v}{2}} \end{cases}$. $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{1}{2\sqrt{2}\sqrt{u+v}} & \frac{1}{2\sqrt{2}\sqrt{u+v}} \\ \frac{1}{2\sqrt{2}\sqrt{u-v}} & -\frac{1}{2\sqrt{2}\sqrt{u-v}} \end{vmatrix}=$
$-\frac{1}{4}\frac{1}{\sqrt{u^2-v^2}}$

(1 pt for choosing new variables u and v. 3 pts for $\frac{\partial(x,y)}{\partial(u,v)}$.)

The corresponding region of D in the uv-plane is a rectangle S which is bounded by $u=3$, $u=4$, $v=1$,

$v = 2$. Hence

$$\iint_D x^3y + 5y^3x \; dA = \iint_D (x^2 + 5y^2)xy \; dA = \iint_S \frac{1}{2}[u + v + 5(u - v)] \cdot \frac{1}{2}\sqrt{u^2 - v^2} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

(2 pts for correct integrand. 1 pt for the region S.)

$$= \frac{1}{16} \int_1^2 \int_3^4 6u - 4v \; dudv = \frac{1}{16} \int_1^2 (3u^2 - 4uv)\Big|_{u=3}^{u=4} dv$$

$$= \frac{1}{16} \int_1^2 21 - 4v \; dv$$ (1 pt for integration w.r.t. u)

$$= \frac{1}{16}(21 - 6) = \frac{15}{16}$$ (1 pt for final answer.)