1. 求以下定積分。 Evaluate the following definite integrals.

   (a) (8%) $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x\sqrt{4-x^2}}\,\mathrm{d}x$ (b) (8%) $\displaystyle\int_0^1 \cos^{-1}(\sqrt{x})\,\mathrm{d}x$

Solution:

(a) Let $x = 2\sin t$. Then $dx = 2\cos t\,dt$ (1%). Since $x \in [1, \sqrt{3}]$, we have that $t \in [\pi/6, \pi/3]$ (1%) and $\cos t > 0$ (1%). Thus

$$\begin{aligned}\int_1^{\sqrt{3}} \frac{dx}{x\sqrt{4-x^2}}\,dx &= \int_{\pi/6}^{\pi/3} \frac{2\cos t\,dt}{2\sin t\cdot 2\cos t} \quad (1\%)\\ &= \frac{1}{2}\int_{\pi/6}^{\pi/3} \csc t\,dt \quad (1\%)\\ &= -\frac{1}{2}\ln|\csc t + \cot t|\Big|_{\pi/6}^{\pi/3} \quad (2\%)\\ &= -\frac{1}{2}\ln\frac{\sqrt{3}}{2+\sqrt{3}} \quad (1\%).\end{aligned}$$

(b) Let $t = \sqrt{x}$. Then $dt = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2t}\,dx$ (1%).

$$\begin{aligned}\int_0^1 \cos^{-1}(\sqrt{x})\,dx &= 2\int_0^1 \cos^{-1}(t)t\,dt \quad (1\%)\\ &= \int_0^1 \cos^{-1}(t)\,dt^2 \quad (1\%)\\ &= \cos^{-1}(t)t^2\Big|_0^1 + \int_0^1 \frac{t^2}{\sqrt{1-t^2}}\,dt \quad (2\%)\\ t := \sin u \qquad &= 0 + \int_0^{\pi/2} \frac{\sin^2 u}{\cos u}\cos u\,du \quad (1\%)\\ &= \int_0^{\pi/2} \frac{1-\cos 2u}{2}\,du \quad (1\%)\\ &= \frac{\pi}{4} \quad (1\%).\end{aligned}$$

2. 求以下不定積分。 Evaluate the following indefinite integrals.

(a) (4%) $\int \frac{x+2}{x^2+4x+5}\,\mathrm{d}x$ (b) (4%) $\int \frac{1}{x^2+4x+5}\,\mathrm{d}x$ (c) (6%) $\int \frac{x-1}{x^2(x^2+4x+5)}\,\mathrm{d}x$

Solution:

(a) Let $\underbrace{u = x^2+4x+5}_{(1M)}$. Then $\underbrace{du = (2x+4)\,dx}_{(1M)}$. Therefore,

$$\int \frac{x+2}{x^2+4x+5}\,\mathrm{d}x = \underbrace{\frac{1}{2}\int \frac{1}{u}\,du}_{(1M)} = \underbrace{\frac{1}{2}\ln(x^2+4x+5)+C}_{(1M)}$$

Remark. Do it by inspection is OK.

(b) $\int \frac{1}{x^2+4x+5}\,\mathrm{d}x = \underbrace{\int \frac{1}{(x+2)^2+1}\,dx}_{(2M)} = \underbrace{\tan^{-1}(x+2)+C}_{(2M)}.$

(c) Suppose $\frac{x-1}{x^2(x^2+4x+5)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4x+5}$ (1M).

By clearing the denominator,

$$(x-1) = Ax(x^2+4x+5)+B(x^2+4x+5)+Cx^3+Dx^2$$

We obtain $\begin{cases} A+C=0 \\ 4A+B+D=0 \\ 5A+4B=1 \\ 5B=-1 \end{cases}$ (1M).

Solving gives $A=\frac{9}{25}$, $B=-\frac{1}{5}$, $C=-\frac{9}{25}$, $D=-\frac{31}{25}$ (0.5 each)

Hence, the given integral equals

$$\begin{aligned}\int \frac{9}{25}\cdot\frac{1}{x}-\frac{1}{5}\cdot\frac{1}{x^2}-\frac{9x+31}{x^2+4x+5}\,dx &= \int \frac{9}{25}\cdot\frac{1}{x}-\frac{1}{5}\cdot\frac{1}{x^2}-\frac{9}{25}\cdot\frac{x+2}{x^2+4x+5}-\frac{13}{25}\cdot\frac{1}{x^2+4x+5}\,dx \\ &= \frac{9}{25}\ln|x|+\frac{1}{5x}-\frac{9}{50}\ln(x^2+4x+5)-\frac{13}{25}\tan^{-1}(x+2)+C\end{aligned}$$

(1M for attempt and 1M for final answer)

3. 計算以下瑕積分或者說明爲什麼它是發散的。
Evaluate each of the following improper integrals, or explain why it is divergent.

(a) (8%) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,\mathrm{d}x$ (b) (4%) $\displaystyle\int_1^2 \frac{1}{x(\ln x)^2}\,\mathrm{d}x$

Solution:

The antiderivative:

$$\int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{(\ln x)^2}\,d(\ln x)$$

Substitution $u = \ln x$

$$= \int u^{-2}\,du = -u^{-1} + C = \frac{-1}{\ln x} + C$$

(a) The improper integral is convergent and it converges to

$$\lim_{b\to\infty} \frac{1}{\ln 2} - \frac{1}{\ln b} = \frac{1}{\ln 2}$$

(b) The improper integral is divergent because

$$\lim_{a\to 1^+} -\frac{1}{\ln 2} + \frac{1}{\ln a} = \frac{-1}{\ln 2} + \lim_{t\to 0^+} \frac{1}{t} = \infty$$

The limit does not exist.

Grading scheme:

(4 pts) Finding the antiderivative. Or converting the improper integral to another one.

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \int_{\ln 2}^\infty \frac{1}{u^2}\,du \quad \text{and} \quad \int_1^2 \frac{1}{x(\ln x)^2}\,dx = \int_0^{\ln 2} \frac{1}{u^2}\,du$$

(4 pts) Explaining why (a) is convergent and evaluating. (-2 pts) if student forgot to evaluate.

(4 pts) Explaining why (b) is divergent.

If the student found the wrong antiderivative or copy the problem wrong, a maximum of (8 pts) is possible. If the problem became too easy to determine convergence/divergence, then the maximum would be (4 pts). The explanation must involve a limit notation, otherwise (-2 pts).

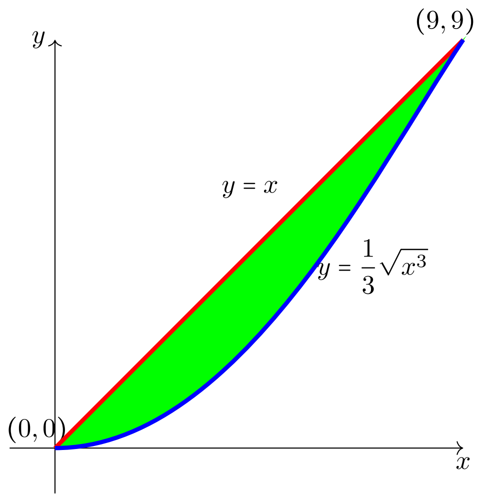
4. 設 R 爲曲線 $y = \frac{1}{3}x^{\frac{3}{2}}$ 和 $y = x$ 在 $x = 0$ 和 $x = 9$ 之間圍成之區域。

Let R be the region enclosed by the curves $y = \frac{1}{3}x^{\frac{3}{2}}$ and $y = x$ between $x = 0$ and $x = 9$.

(a) (8%) 求 R 繞 x-軸旋轉的旋轉體體積。
Find the volume of the solid of revolution obtained by rotating R about the x-axis.

(b) (8%) 求 R 繞 y-軸旋轉的旋轉體體積。
Find the volume of the solid of revolution obtained by rotating R about the y-axis.

(c) (8%) 求 R 的總周長。（包括曲線和直線部份）
Find the perimeter of R (that is, the combined length of the two arcs).



Solution:

(a) The integral that evaluates the volume can be set up in two ways.

$$\int_0^9 \left[\pi(x)^2 - \pi\left(\frac{1}{3}x^{3/2}\right)^2\right] dx = \int_0^9 2\pi(y)\left[(3y)^{2/3} - y\right] dy$$

We can evaluate either.

$$\int_0^9 \left[\pi(x)^2 - \pi\left(\frac{1}{3}x^{3/2}\right)^2\right] dx = \frac{\pi}{9}\int_0^9 \left(9x^2 - x^3\right) dx = \frac{\pi}{9}\left[3x^3 - \frac{x^4}{4}\right]_0^9 = \frac{243\pi}{4}$$

(b) The integral that evaluates the volume can be set up in two ways.

$$\int_0^9 2\pi(x)\left(x - \frac{1}{3}x^{3/2}\right) dx = \int_0^9 \left[\pi\left((3y)^{2/3}\right)^2 - \pi(y)^2\right] dy$$

We can evaluate either.

$$\int_0^9 2\pi(x)\left(x - \frac{1}{3}x^{3/2}\right) dx = \frac{2\pi}{3}\int_0^9 \left(3x^2 - x^{5/2}\right) = \frac{2\pi}{3}\left[x^3 - \frac{2}{7}x^{7/2}\right]_0^9 = \frac{486\pi}{7}$$

(c) The length of the line segment from $(0,0)$ to $(9,9)$ is $9\sqrt{2}$.

The length of the curve $y = \frac{1}{3}x^{3/2}$ is given by

$$\int_0^9 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^9 \sqrt{1 + \frac{x}{4}}\, dx = \frac{1}{3}\left[(x+4)^{3/2}\right]_0^9 = \frac{13\sqrt{13} - 8}{3}$$

The combined length is $\frac{1}{3}(13\sqrt{13} + 27\sqrt{2} - 8)$.

Grading scheme:

(4 pts) for each correct integral setup. (-4 pts) if they confuse (a) and (b).

Students do not lose points if they notice a negative answer and add a negative sign to fix it. (-4 pts) for each negative volume or length.

(-2 pts) for each computational mistake. (-1 pt) if they over-simplified.

(-2 pts) if students forgot $9\sqrt{2}$.

5. (a) (4%) 寫出 $\dfrac{1}{\sqrt{1-x^2}}$ 在 $x = 0$ 的泰勒展開式。(你可以用 C_r^a 這個記號表示答案。)

Write down the Taylor series of $\dfrac{1}{\sqrt{1-x^2}}$ at $x = 0$. (You may express your answer in terms of C_r^a notation.)

(b) (6%) 假設 $\sin^{-1} x$ 在 $x = 0$ 的泰勒展開式爲：

$$\sin^{-1} x = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \cdots$$

寫出 a_1, a_3 和 a_5 的值。(答案請用分數表示)

Suppose the Taylor expansion of $\sin^{-1} x$ at $x = 0$ is

$$\sin^{-1} x = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \cdots$$

Write down the values of a_1, a_3 and a_5 as a fraction.

Solution:

(a) By the binomial theorem, we have

$$\begin{aligned}(1-x^2)^{-1/2} &= C_0^{-1/2} + C_1^{-1/2}(-x^2) + C_2^{-1/2}(-x^2)^2 + \cdots \\ &= C_0^{-1/2} - C_1^{-1/2}x^2 + C_2^{-1/2}x^4 - \cdots \quad (4\%).\end{aligned}$$

(正負號打錯的話扣一分, x的次方打錯扣一分)

(b) We have that

$$\begin{aligned}\sin^{-1} x &= \int_0^x (1-t^2)^{-1/2}\,dt \quad (2\%) \\ &= C_0^{-1/2}x - C^{-1/2} \cdot \frac{x^2}{3} + C_2^{-1/2} \cdot \frac{x^5}{5} - \cdots \quad (1\%) \\ &= x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots.\end{aligned}$$

Hence $a_1 = 1(1\%)$, $a_3 = \dfrac{1}{6}(1\%)$ and $a_5 = \dfrac{3}{40}(1\%)$.

6. 計算以下極限。 Evaluate the following limits.

(a) (8%) $\lim_{x\to 0^+} \frac{1}{\sqrt{x}} \int_{-\sqrt{x}}^{\sqrt{x}} e^{-t^2}\,\mathrm{d}t.$

(b) (8%) $\lim_{x\to 0} \frac{e^{-x^3} - 1 + x^3}{x^6}.$

(提示：先寫出 e^{-x^3} 在 $x = 0$ 的泰勒展開式。) (`Hint.` Write down the Taylor series of e^{-x^3} at $x = 0$)

(c) (8%) $\lim_{x\to 0} (\cos x)^{\frac{1}{x}}$

Solution:

(a)

$$\lim_{x\to 0^+} \frac{\int_{-\sqrt{x}}^{\sqrt{x}} e^{-t^2}\,dt}{\sqrt{x}} \overset{(0/0)}{=} \lim_{x\to 0^+} \frac{e^{-x}\cdot\frac{1}{2\sqrt{x}} - e^{-x}\cdot(-\frac{1}{2\sqrt{x}})}{\frac{1}{2\sqrt{x}}}$$
$$= \lim_{x\to 0^+} 2e^{-x}$$
$$= 2$$

Marking scheme for (a)

- 1M in correctly identifying the indeterminate form
- 1M for using L'Hospital's rule
- 4M for correct derivative of numerator (2M for applying FTC, 1M for the term from chain rule, 1M for the term arisen from the lower bound)
- 1M for correct derivative of denominator
- 1M for correct answer

(b) Since $e^x = 1 + x + \frac{x^2}{2!} + \cdots$, we have $e^{-x^3} = 1 - x^3 + \frac{x^6}{2} + \cdots$ Hence,

$$\lim_{x\to 0} \frac{e^{-x^3} - 1 + x^3}{x^6} = \lim_{x\to 0} \frac{(1 - x^3 + \frac{x^6}{2} + \cdots) - 1 + x^3}{x^6}$$
$$= \lim_{x\to 0} \frac{1}{2} + \cdots$$
$$= \frac{1}{2}$$

Marking scheme for (b)

- 2M for knowing the series expansion for e^x
- 2M for the correct series expansion for e^{-x^3}
- 2M for simplifying the fraction by plugging in the series
- 2M for correct answer

We accept doing (b) by L'Hospital-ing six times. In that case, 1M for each correct application of L'H and the final 2M for the correct answer.

(c) Let $y = (\cos x)^{\frac{1}{x}}$. Then $\ln y = \frac{\ln(\cos x)}{x}$. Therefore,

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(\cos x)}{x}$$
$$\overset{(0/0)}{=} \lim_{x\to 0^+} \frac{-\tan x}{1}$$
$$= 0$$

Hence, $\lim_{x\to 0} (\cos x)^{\frac{1}{x}} = e^0 = 1$.

Marking scheme for (c)

- 2M for taking log
- 1M in correctly identifying the indeterminate form
- 1M for correct derivative of numerator
- 1M for correct derivative of denominator
- 2M for correct limit of $\ln y$
- 1M for correct answer