1. Consider $F(x) = \int_{\frac{1}{x}}^{x} \sin(\sqrt{xt})\,\mathrm{d}t$.

   (a) (3%) Show that $F(x) = \frac{1}{x} \cdot \int_{1}^{x^2} \sin(\sqrt{u})\,\mathrm{d}u$.

   (b) (7%) By using (a), find $F'(1)$.

Solution:

Marking Scheme for 1(a)

- 2M for letting $u = xt$
- 1M for completing the proof

Sample Solution for 1(a).

Let $u = xt$. (2M) Then $du = xdt \Rightarrow \frac{1}{x}du = dt$ and $\begin{cases} \text{when } t = \frac{1}{x},\ u = 1 \\ \text{when } t = x,\ u = x^2 \end{cases}$ (1M) Therefore,

$$F(x) = \int_{\frac{1}{x}}^{x} \sin(\sqrt{xt})\,\mathrm{d}t = \int_{1}^{x^2} \sin(\sqrt{u}) \cdot \frac{1}{x}\,dt = \frac{1}{x} \cdot \int_{1}^{x^2} \sin(\sqrt{u})\,\mathrm{d}u$$

Marking Scheme for 1(b)

- 2M for the use of product/L'Hospital's rule
- 3M for differentiating the integral correctly (-2M for missing the term from chain rule)
- 1M for $F(1) = 0$
- 1M for answer

Sample Solution I for 1(b).

By product rule (2M), $F'(x) = -\frac{1}{x^2} \cdot \int_{1}^{x^2} \sin(\sqrt{u})du + \frac{1}{x} \cdot \underbrace{\sin(x) \cdot 2x}_{(3M)}$. Therefore,

$$F'(1) = -1 \cdot \underbrace{0}_{(1M)} + \sin(1) \cdot 2 = \underbrace{2\sin(1)}_{(1M)}$$

Sample Solution II for 1(b).

$$\begin{aligned}
F'(1) &\overset{\text{def}}{=} \lim_{h\to 0} \frac{F(1+h) - F(1)}{h} = \lim_{h\to 0} \frac{F(1+h) - 0}{h} && \text{(1M) for } F(1) = 0 \\
&= \lim_{h\to 0} \frac{\int_{1}^{(1+h)^2} \sin(\sqrt{u})\,du}{h^2 + h} \\
&= \lim_{h\to 0} \frac{\sin(1+h) \cdot 2(h+1)}{2h+1} && \text{(2M) for L'H and (3M) for derivative} \\
&= 2\sin(1) && \text{(1M) for answer}
\end{aligned}$$

2. Evaluate the following integrals.

(a) (9%) $\int (x+3)\sqrt{1-x^2}\,dx$.

(b) (9%) $\int_0^1 \ln(\sqrt{x}+2)\,dx$.

Solution:

(a) Let $x = \sin\theta$ with θ between $-\pi/2$ and $\pi/2$. So, $dx = \cos\theta d\theta$ and

$$\begin{aligned}
&\int (x+3)\sqrt{1-x^2}dx \\
&= \int (\sin\theta + 3)\cos^2\theta d\theta \quad \text{(3\%)} \\
&= -\frac{1}{3}\cos^3\theta + 3\int \frac{1+\cos 2\theta}{2} d\theta \\
&= -\frac{1}{3}\cos^3\theta + \frac{3}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C \quad \text{(3\%)} \\
&= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + \frac{3}{2}\sin^{-1}x + \frac{3}{2}x\sqrt{1-x^2} + C. \quad \text{(3\%)}
\end{aligned}$$

(b) Let $u = \sqrt{x}$. So, $2udu = dx$ and

$$\begin{aligned}
&\int_0^1 \ln(\sqrt{x}+2)dx = \int_0^1 \ln(u+2)2udu \quad \text{(2\%)} \\
&= \ln(u+2)u^2\Big|_0^1 - \int_0^1 u^2\frac{1}{u+2}du = \ln 3 - \int_0^1 \frac{u^2}{u+2}du. \quad \text{(3\%)}
\end{aligned}$$

For the second term,

$$\begin{aligned}
&\int_0^1 \frac{u^2}{u+2}du = \int_0^1 u - 2 + \frac{4}{u+2}du \quad \text{(2\%)} \\
&= \frac{u^2}{2} - 2u + 4\ln|u+2|\Big|_0^1 = -\frac{3}{2} + 4\ln 3 - 4\ln 2.
\end{aligned}$$

Therefore, the final answer is

$$\ln 3 + \frac{3}{2} - 4\ln 3 + 4\ln 2 = \frac{3}{2} - 3\ln 3 + 4\ln 2. \quad \text{(2\%)}$$

3. (a) (8%) Evaluate $\int \cot^3(x)\,\mathrm{d}x$ and apply the result to show $\int_0^{\pi/2} \cot^3(x)\,\mathrm{d}x$ is divergent.

(b) (4%) Use the *Comparison Theorem for Improper Integral* to determine whether

$$\int_1^{\infty} \frac{1}{e^x - 2^x}\,\mathrm{d}x$$

is convergent or divergent.

Solution:

(a) The following crucial steps must be shown clearly ⋯

4%⋯ $$\begin{aligned}\int \cot^3(x)\,dx &= \int \cot(x)(\csc^2(x) - 1)\,dx \\ &= -\int \cot(x)\cot'(x)dx - \int \cot(x)\,dx \\ &= -\frac{\cot^2(x)}{2} - \ln|\sin(x)| + C\end{aligned}$$

4%⋯ $$\begin{aligned}\int_0^{\pi/2} \cot^3(x)\,dx &= \lim_{a\to 0^+} (\frac{\cot^2(a)}{2} + \ln|\sin(a)|) \\ &= \lim_{a\to 0^+} \frac{1}{2\sin^2(a)}(1 + \sin^2(a)\ln|\sin(a)|) \\ &= \lim_{a\to 0^+} \frac{1}{2\sin^2(a)}(1 + 0) = \infty\end{aligned}$$

(b) The following crucial steps must be shown clearly ⋯

1%⋯ $$e^x - 2^x = e^x(1 - \frac{2^x}{e^x}) = e^x(1 - (\frac{2}{e})^x)$$

1%⋯ $$\lim_{x\to\infty} (\frac{2}{e})^x = 0 \Rightarrow (\frac{2}{e})^x < 1/2 \text{ whenever } x > b = \frac{\ln(2)}{1 - \ln(2)}$$

2%⋯ $$\begin{aligned}\int_1^{\infty} \frac{dx}{e^x - 2^x} &= \int_1^{b} \frac{dx}{e^x - 2^x} + \int_b^{\infty} \frac{dx}{e^x - 2^x} \\ &< \int_1^{b} \frac{dx}{e^x - 2^x} + \int_b^{\infty} \frac{2dx}{e^x} = 2e^{-b} + \int_1^{b} \frac{dx}{e^x - 2^x} < \infty\end{aligned}$$

convergent

4. (a) (9%) Find the orthogonal trajectories of the family of curves $y^2 = 4 - Cx$, where C is an arbitrary constant.
   (b) (9%) Solve, for $y = f(x)$, the equation

$$\frac{1}{x} \cdot \frac{dy}{dx} - 2y = e^{x^2} \sin(x)\cos(x) \text{ with } f\left(\frac{\pi}{3}\right) = 0.$$

Solution:

(a) Differentiating the given equation, we get

$$2ydy = -Cdx$$

or

$$\frac{dy}{dx} = -\frac{C}{2y}.$$

Solving the given equation for C ,

$$C = \frac{4 - y^2}{x}.$$

Plugging this in the differential equation to get rid of C from the expression of $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{y^2 - 4}{2xy}.$$

Another equivalent method to find $\frac{dy}{dx}$ in terms of x, y is as follows: First, solve the given equation for C:

$$C = \frac{4 - y^2}{x}.$$

Then differentiate both sides to get

$$0 = \frac{d}{dx}\left(\frac{4 - y^2}{x}\right) = \frac{-2y\frac{dy}{dx}x - (4 - y^2)}{x^2}.$$

This is the differential equation whose solutions give the given family of curves. By simplification, we get $\frac{dy}{dx} = \frac{y^2 - 4}{2xy}$.

The next step is to set up the differential equation for the orthogonal trajectories:

$$\frac{dy}{dx} = -\frac{1}{\frac{y^2-4}{2xy}} = -\frac{2xy}{y^2 - 4}.$$

Separating the variables, we have

$$\frac{y^2 - 4}{2y}dy = -xdx.$$

Thus,

$$\int \frac{y^2 - 4}{2y}dy = -\int xdx$$

by which we get

$$\frac{y^2}{4} - 2\ln|y| = -\frac{x^2}{2} + C.$$

(a) Marking scheme

- 4 pts for finding the differential equation

$$2xy\frac{dy}{dx} = y^2 - 4$$

  (or any equivalent form) of the given family of curves in x, y. Partial credits will be:
  - 3 pts, if the student seems to try eliminating C, but incorrectly.
  - 2 pts, if the differential equation is given correctly, but containing C.

- 1 pts, if the differential equation is given incorrectly, and containing C.
- 2 pts for setting up the differential equation of the orthogonal trajectories

$$\frac{dy}{dx} = -\frac{2xy}{y^2 - 4}.$$

Partial credits will be:

  - 1 pt, if the student tries to set up the equation but incorrectly, such as

$$\frac{dy}{dx} = \frac{2xy}{y^2 - 4} \text{ or } \frac{dy}{dx} = -\frac{y^2 - 4}{2xy} \text{ etc.}$$

- 3 pts for solving the differential equation of the orthogonal trajectories.
  - 1 pt for separating the equation into the terms of x and those of y.
  - 2 pts for writing down the final equation correctly.
    * 1 pt is taken away if there are minor mistakes such as missing the absolute value sign of the logarithm.
  - 1 pt for executing the integration correctly, but integrating incorrect integral caused by the previous mistakes.

(b) Making the given equation into the standard form, we have

$$\frac{dy}{dx} - 2xy = xe^{x^2} \sin x \cos x.$$

Since

$$\int (-2x)dx = -x^2 + C$$

so we can choose the integration factor to be $I(x) = e^{-x^2}$. Multiplying e^{-x^2} through,

$$\frac{d}{dx}\left(e^{-x^2} y\right) = x \sin x \cos x.$$

Integrating both sides, we get

$$\begin{aligned} e^{-x^2} y &= \int x \sin x \cos x dx \\ &= \frac{1}{2} \int x \sin 2x dx \\ &= \frac{1}{2} x \left(-\frac{\cos 2x}{2}\right) - \frac{1}{2} \int \left(-\frac{\cos 2x}{2}\right) dx \\ &= -\frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + C. \end{aligned}$$

Hence,

$$y = e^{x^2} \left(-\frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + C\right).$$

Using the initial condition

$$f\left(\frac{\pi}{3}\right) = 0,$$

we get

$$-\frac{1}{4} x \cos\left(\frac{2\pi}{3}\right) + \frac{1}{8} \sin\left(\frac{2\pi}{3}\right) + C = 0.$$

Hence

$$C = \frac{\pi}{24} - \frac{\sqrt{3}}{16}.$$

(b) Marking scheme

- 4 pts for finding the correct "integrable" form

$$\frac{d}{dx}\left(e^{-x^2} y\right) = x \sin x \cos x$$

- 3 pts for finding integration factor $I(x) = e^{-x^2}$ correctly.
    - 1 pt for rewriting the equation in the standard form.

- 3 pts for executing the integration.
  - 2 pts for the student tries integration by parts,
    - 1 pt if the student just struggles, but gets to nowhere
- 2 pts for determining the constant C using

$$f\left(\frac{\pi}{3}\right) = 0.$$

  It's not necessary to write down the final equation, if the constant is determined correctly.
  - 1 pts for setting up the equation for the constant.

If the student set up an incorrect equation, but execute the remaining calculations correctly, he/she will get 1 pt for the correct integration, 1 pt for setting up the equation for determining the constant (so at most 2 pts after setting up the integrable form).

5. (14%) Consider the region enclosed by the curve $y = \dfrac{5}{x\sqrt{5-x}}$, the x-axis, the lines $x = 1$, $x = 4$ in the first quadrant. Find the volume of the solid obtained by rotating this region about the x-axis.

Solution:

The volume is

$$V = \int_1^4 \pi \left(\frac{5}{x\sqrt{5-x}} \right)^2 dx = \pi \int_1^4 \frac{-25}{x^2(x-5)}\, dx \ (4\%)$$

$$\frac{-25}{x^2(x-5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-5} \ (3\%)$$

This shows us that

$$-25 = Ax(x-5) + B(x-5) + Cx^2 = (A+C)x^2 + (B-5A)x - 5B$$

So we get

$$B = 5, \quad A = 1, \quad C = -1. \ (3\%)$$

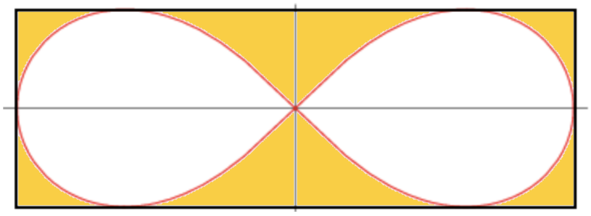
Hence,

$$\begin{aligned} V &= \pi \left(\int_1^4 \frac{1}{x}\, dx + \int_1^4 \frac{5}{x^2}\, dx - \int_1^4 \frac{1}{x-5}\, dx \right) \\ &= \pi \left(\ln|x| - \frac{5}{x} - \ln|x-5| \Big]_1^4 \right) = \left(4\ln 2 + \frac{15}{4} \right) \pi \ (4\%) \end{aligned}$$

6. Consider a lemniscate C whose polar equation is given by $r^2 = \cos(2\theta)$.

(a) (6%) Find all the points (in polar coordinates) of C at which the tangent to C is horizontal.

(b) (8%) The curve C is inscribed in a rectangle such that each side of the rectangle is tangent to C (see figure below). Find the area of the shaded region.



Solution:

Marking Scheme for 6(a)

- 1M for realizing that $\frac{dy}{d\theta} = 0$
- 1M for writing $y = \sqrt{\cos(2\theta)} \cdot \sin\theta$
- 1M for correct computation of $\frac{dy}{d\theta}$
- 0.5M for each correct angular coordinates θ
- 1M for the correct radical coordinate r

Sample Solution for 6(a).

Since $\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)}$, a tangent is horizontal is equivalent to having $\frac{dy}{d\theta} = 0$. (1M)

Since $y = r\sin\theta = \sqrt{\cos(2\theta)} \cdot \sin\theta$, (1M)

we have $\frac{dy}{d\theta} = \frac{-\sin(2\theta)}{\sqrt{\cos(2\theta)}} \cdot \sin\theta + \sqrt{\cos(2\theta)} \cdot \cos\theta$. (1M)

So $\frac{dy}{d\theta} = 0$ implies $\tan(2\theta) \cdot \tan\theta = 1$. Thus, we have $\frac{2\tan\theta}{1 - \tan^2\theta} \cdot \tan\theta = 1 \Rightarrow \tan\theta = \pm\frac{1}{\sqrt{3}} \Rightarrow \theta = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$. (2M)

In all cases, $r = \frac{1}{\sqrt{2}}$. (1M)

Therefore, the four required points are $(r, \theta) = \left(\frac{1}{\sqrt{2}}, \pm\frac{\pi}{6}\right)$ and $\left(\frac{1}{\sqrt{2}}, \pm\frac{5\pi}{6}\right)$

Marking Scheme for 6(b)

- 2M for the correct integrand for the area of lemniscate
- 2M for the correct integration limits for the area of lemniscate
- 1M for the correct area of lemniscate
- 1M for the correct height of the rectangle OR y-coordinate of the highest point of lemniscate
- 1M for the correct area of rectangle
- 1M for the correct final answer

Sample Solution for 6(b).

First we calculate the area enclosed by the lemniscate C :

$$4 \cdot \underbrace{\int_0^{\frac{\pi}{4}}}_{2M} \underbrace{\frac{1}{2}\cos(2\theta)}_{2M}\, d\theta = [\sin(2\theta)]_0^{\frac{\pi}{4}} = \underbrace{1}_{(1M)}$$

The Cartesian coordinate of $(r,\theta) = \left(\frac{1}{\sqrt{2}}, \pm\frac{\pi}{6}\right)$ is $(x,y) = \left(\frac{\sqrt{3}}{2\sqrt{2}}, \underbrace{\frac{1}{2\sqrt{2}}}_{\text{(1M)}}\right)$. Therefore, the area of the rectangle equals

$$2 \cdot \frac{1}{\sqrt{2}} = \underbrace{\sqrt{2}}_{\text{(1M)}}$$

Hence, the area of the shaded region equals $\sqrt{2} - 1$ (1M).

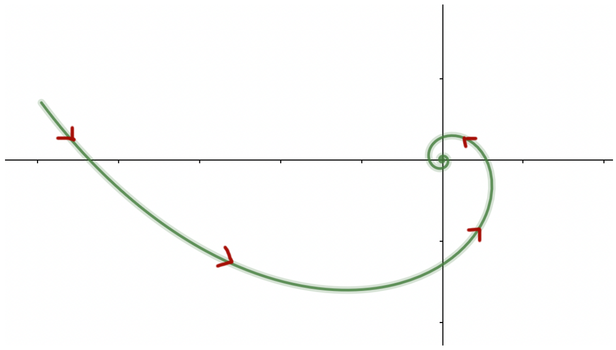
7. For each real number p, the parametric equations

$$x(t) = \frac{\cos(3t)}{t^p}, \ y(t) = \frac{\sin(3t)}{t^p} \text{ with } 1 \le t < \infty$$

define an *improper spiral*.

(a) (12%) Find the arclength of the *improper spiral* when $p = 3$ (see figure).

(b) (2%) Find the range of values of p such that the arclength of the *improper spiral* is finite.



Solution:

(a) $p = 3$.

$$x(t) = \frac{\cos(3t)}{t^3} \ , \ y(t) = \frac{\sin(3t)}{t^3} \quad \text{with} \quad 1 \le t < \infty$$

$$x'(t) = \frac{-3\sin(3t)}{t^3} - \frac{3\cos(3t)}{t^4} \ , \ y'(t) = \frac{3\cos(3t)}{t^3} - \frac{3\sin(3t)}{t^4}$$

$$[x'(t)]^2 + [y'(t)]^2 = \frac{9}{t^6} + \frac{9}{t^8}$$

$$\int_1^\infty \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \lim_{a\to\infty} \int_1^a \frac{3\sqrt{t^2+1}}{t^4}\, dt = \lim_{b\to\frac{\pi}{2}^-} \int_{\pi/4}^b \frac{3\sec^3\theta}{\tan^4\theta}\, d\theta$$

$$= \lim_{b\to\frac{\pi}{2}^-} \int_{\pi/4}^b \frac{3\cos\theta}{\sin^4\theta}\, d\theta = \lim_{b\to\frac{\pi}{2}^-} \left[-(\sin\theta)^{-3}\right]_{\pi/4}^b = 2\sqrt{2} - 1$$

(b)

$$x'(t) = \frac{-3\sin(3t)}{t^p} - \frac{p\cos(3t)}{t^{p+1}} \ , \ y'(t) = \frac{3\cos(3t)}{t^p} - \frac{p\sin(3t)}{t^{p+1}}$$

$$[x'(t)]^2 + [y'(t)]^2 = \frac{9}{t^{2p}} + \frac{p^2}{t^{2p+2}}$$

The arc length is

$$\int_1^\infty \sqrt{\frac{9}{t^{2p}} + \frac{p^2}{t^{2p+2}}}\, dt = \int_1^\infty \frac{1}{t^p}\sqrt{9 + \frac{p^2}{t^2}}\, dt$$

Since $3 < \sqrt{9 + \frac{p^2}{t^2}} < \sqrt{9 + p^2}$ bounded by constants, we can compare with $\int_1^\infty \frac{1}{t^p}\, dt$.

Therefore the arc length is finite when $p > 1$.

Grading scheme:

(2 pts) for arc length formula.

(8 pts) for integration in (a). (4 pts) for trig-sub and (4 pts) for trig-integral.

(2 pts) for notation and answer in (a).

(1 pt) for setting up the improper integral with p in (b).

(1 pt) for range of p.