1. (20 pts) 計算以下的極限。(不可以使用 L'Hospital's rule)
   Evaluate the following limits. (Use of L'Hospital's rule is not allowed.)

   你可以直接使用以下的極限 （不用證明）。
   You may use, without proof, the following standard limits.

   $$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x=e \qquad \lim_{x\to 1}\frac{\ln x}{x-1}=1$$

   (a) (5 pts) $\lim_{x\to\infty}\frac{3^x+e^x}{1+2^{2x}}$ (b) (5 pts) $\lim_{x\to 0}\frac{1-\cos(2x)}{\sqrt{x^2+4}-2}$ (c) (5 pts) $\lim_{x\to\frac{\pi}{2}}\frac{\ln(\sin^2 x)}{\cos^2 x}$ (d) (5 pts) $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$

Solution:

(a)

$$\begin{aligned}\lim_{x\to\infty}\frac{3^x+e^x}{1+2^{2x}} &= \lim_{x\to\infty}\frac{1^x+(e/3)^x}{(1/3)^x+(4/3)^x} \quad (3\%)\\ &= 0 \quad (2\%).\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos(2x)}{\sqrt{x^2+4}-2} &= \lim_{x\to 0}\frac{1-\cos(2x)}{\sqrt{x^2+4}-2}\,\frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2}\\ &= \lim_{x\to 0}\frac{(1-\cos(2x))(\sqrt{x^2+4}+2)}{x^2} \quad (2\%)\\ &= \lim_{x\to 0}\frac{2\sin^2 x}{x^2}\left(\sqrt{x^2+4}+2\right) \quad (2\%)\\ &= 2\cdot 4 = 8 \quad (1\%).\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to\pi/2}\frac{\ln(\sin^2 x)}{\cos^2 x} &= -\lim_{x\to\pi/2}\frac{\ln(\sin^2 x)}{\sin^2 x-1} \quad (2\%)\\ (\text{let } t=\sin^2 x)\qquad &= -\lim_{t\to 1}\frac{\ln t}{t-1} \quad (2\%)\\ &= -1 \quad (1\%).\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x &= \lim_{x\to\infty}\left(1+\frac{1}{(x-1)/2}\right)^{2\cdot\frac{x-1}{2}+1} \quad (3\%)\\ (\text{let } y=(x-1)/2)\qquad &= \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{2y+1}\\ &= e^2\cdot 1 = e^2 \quad (2\%).\end{aligned}$$

2. (20 pts) 計算以下導函數或導數。 Compute the following derivatives.

(a) (8 pts) $f(x) = x^3 \cdot e^{(x^2+2)}$, 求 $f'(x)$ 和 $f''(x)$。Find $f'(x)$ and $f''(x)$.

(b) (6 pts) $g(x) = \dfrac{x}{\tan^{-1}(\sin x)}$, 求 $g'(x)$。Find $g'(x)$.

(c) (6 pts) $h(x) = x^x \cdot (1+x^2)^{\frac{1}{x}}$, 求 $h'(1)$。 Find $h'(1)$.

Solution:

(a) Product rule and chain rule.

$$f'(x) = \left(3x^2\right) \cdot e^{x^2+2} + \left(e^{x^2+2} \cdot 2x\right) \cdot x^3 = \left(2x^4 + 3x^2\right) e^{x^2+2}$$

$$f''(x) = \left(8x^3 + 6x\right) \cdot e^{x^2+2} + \left(e^{x^2+2} \cdot 2x\right) \cdot \left(2x^4 + 3x^2\right) = \left(4x^5 + 14x^3 + 6x\right) e^{x^2+2}$$

(b) Quotient rule and chain rule.

$$g'(x) = \frac{(1) \cdot \tan^{-1}(\sin x) - \left(\frac{\cos x}{1+\sin^2 x}\right) \cdot x}{\left(\tan^{-1}(\sin x)\right)^2}$$

(c) Logarithmic differentiation.

$$\ln h(x) = x \ln x + \frac{\ln(1+x^2)}{x}$$

$$\frac{h'(x)}{h(x)} = \ln x + 1 + \frac{2}{1+x^2} - \frac{\ln(1+x^2)}{x^2}$$

$h'(1) = h(1) \cdot (0 + 1 + 1 - \ln 2) = 4 - 2\ln 2.$

Grading scheme:

Part (a) is 5+3. Part (b) is 6. Part (c) is 5+1.

0 points if they couldn't use the derivative rules correctly.

-3 points if they clearly remembered a derivative formula wrong.

-1 point for computational mistakes/miscopy/oversimplify.

3. (12 pts) 方程 $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ 在點 $\left(0, \frac{1}{2}\right)$ 附近可以描寫成隱函數 $y = y(x)$。

Near the point $\left(0, \frac{1}{2}\right)$, the equation $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ defines implicitly a function $y = y(x)$.

(a) (7 pts) 求導數 $\left.\frac{dy}{dx}\right|_{(0,\frac{1}{2})}$。 Find $\left.\frac{dy}{dx}\right|_{(0,\frac{1}{2})}$.

(b) (5 pts) 使用 $y(x)$ 在 $x = 0$ 的線性逼近去估算 $y(0.1)$ 的值。
Use linear approximation of $y(x)$ at $x = 0$ to approximate the value of $y(0.1)$.

Solution:

(a) Regard y as a function of x and take derivatives on both sides of the equality above we have

$$2x + 2y\frac{dy}{dx} = 2(2x^2 + 2y^2 - x) \cdot (4x + 4y\frac{dy}{dx} - 1) \quad (4\%)$$

Put $(x, y) = (0, 1/2)$ in the above equality, we have

$$\left.\frac{dy}{dx}\right|_{(0,1/2)} = 2 \cdot 2 \cdot \frac{1}{4}\left(2\left.\frac{dy}{dx}\right|_{(0,1/2)} - 1\right) \quad (2\%)$$
$$\Rightarrow \left.\frac{dy}{dx}\right|_{(0,1/2)} = 1 \quad (1\%).$$

(b)

$$y(x) \approx y(0) + y'(0)x \quad (2\%) \Rightarrow y(0.1) \approx \frac{1}{2} + 1 \cdot 0.1 = 0.6 \quad (3\%).$$

4. (10 pts) 考慮函數 Consider the function $f(x) = 4\tan^{-1}(x^3) + e^x$.

(a) (4 pts) 說明：$f(x)$ 的反函數存在。 Explain briefly why the inverse function of $f(x)$ exists.

(b) (6 pts) 設 $g(x)$ 爲 $f(x)$ 的反函數， 求 $g'(\pi + e)$。 Let $g(x)$ be the inverse function of $f(x)$. Find $g'(\pi + e)$.

Solution:

(a)
- (2M) Correct $f'(x)$
- (1M) Writing $f'(x) > 0$
- (1M) Saying $f(x)$ is strictly increasing

Sample solution.

$$f'(x) = \underbrace{\frac{12x^2}{1+x^6} + e^x}_{2M} \underbrace{> 0}_{1M}$$

so f is strictly increasing ⋯(1M).
This implies that the inverse function of $f(x)$ exists.

(b)
- (1M) Discovering $f(1) = \pi + e$
- (2M) Correct $f'(1)$
- (1M) Correct formula for $g'(f(x))$
- (2M) Correct answer

Sample solution.

Note $f(1) = \pi + e$ ⋯ (1M)
and $f'(1) = 6 + e$ ⋯ (2M).

$$\text{Therefore, } g'(\pi + e) = g'(f(1)) = \frac{1}{f'(1)} \cdots \text{ (1M)}$$
$$= \frac{1}{6+e} \cdots \text{ (2M)}$$

5. (6 pts) 說明：$x = 0$ 爲方程式 $x + \cos^{-1} x = \frac{\pi}{2}$ 的唯一解。

Prove that $x = 0$ is the only solution to the equation : $x + \cos^{-1} x = \frac{\pi}{2}$.

Solution:

- 2M - Correct citing of Rolle/MVT/Consequences of MVT
- 2M - Correct derivative of $x + \cos^{-1}(x)$
- 2M - Overall correct and complete argument

Sample solution 1.
Let $F(x) = x + \arccos(x)$. Suppose $x = \alpha$ is another solution to the equation. Then $F(0) = F(\alpha)$.
Hence, Rolle's Theorem implies that $F'(c) = 0$ for some c lying strictly between 0 and α. $\cdots$ (2M)
However, $F'(c) = 0$ implies $\underbrace{1 - \frac{1}{\sqrt{1-c^2}}}_{2\text{M}} = 0$ and
hence $c = 0$ which is a contradiction. $\cdots$ (Complete, correct argument 2M)

Sample solution 2.
Let $F(x) = x + \arccos(x)$.
Then $\underbrace{1 - \frac{1}{\sqrt{1-x^2}}}_{2\text{M}} < 0$ for $-1 < x < 0$ and $0 < x < 1$.
Therefore, F is strictly decreasing before reaching $x = 0$ and also after leaving $x = 0$.$\cdots$ (2M)
Hence the function crosses $y = \frac{\pi}{2}$ at most (and hence exactly) once. $\cdots$ (Complete, correct argument 2M)

6. (20 pts) 考慮函數 Consider the function $f(x) = 6x - x^2 - 4\ln x$.

(a) (1 pt) 寫出函數 $f(x)$ 的定義域。 Write down the domain of $f(x)$.

(b) (4 pts) 求 $f'(x)$，找出函數 $f(x)$ 遞增、遞減的區間。
Find $f'(x)$. Write down the interval(s) of increase and interval(s) of decrease of $f(x)$.

(c) (4 pts) 求 $f''(x)$， 判斷 $y = f(x)$ 的凹性。
Find $f''(x)$. Write down the interval(s) on which $f(x)$ is concave upward and the interval(s) on which $f(x)$ is concave downward.

(d) (4 pts) 找出所有局部最大/小値和反曲點。 Write down (if any) the local extremas and inflection points.

(e) (2 pts) 找出 $y = f(x)$ 的所有漸近線。 Find all the asymptotes of $y = f(x)$.

(f) (5 pts) 畫出 $y = f(x)$ 的圖形。 Sketch the graph of $y = f(x)$.

Solution:

(a) $x > 0$.

(b)

$$f'(x) = 6 - 2x - \frac{4}{x} = \frac{-2(x^2 - 3x + 2)}{x} = \frac{-2(x-1)(x-2)}{x}$$

Increasing: $(1, 2)$

Decreasing: $(0, 1)$ and $(2, \infty)$

(c)

$$f''(x) = -2 + \frac{4}{x^2} = \frac{-2(x^2 - 2)}{x^2} = \frac{-2(x - \sqrt{2})(x + \sqrt{2})}{x^2}$$

Concave upward: $(0, \sqrt{2})$

Concave downward: $(\sqrt{2}, \infty)$

(d) Local minimum at $x = 1, y = 5$, local maximum at $x = 2, y = 8 - 4\ln 2$, inflection point $(\sqrt{2}, 6\sqrt{2} - 2 - 2\ln 2)$.

(e) Vertical asymptote at $x = 0$ since $\lim_{x\to 0^+} f(x) = \infty$.

No horizontal asymptote as x goes to infinity since $\lim_{x\to\infty} f(x) = -\infty$.

No slant asymptote as x goes to infinity since $\lim_{x\to\infty} \frac{f(x)}{x} = -\infty$.

(f)

Grading scheme:

(a) (1pts) Right 1 or wrong 0.

(b) (5pts) 2 pts for derivative. 3 pts for determining the signs.

(c) (5pts) Follow through. 2 pts for derivative. 3 pts for determining the signs.

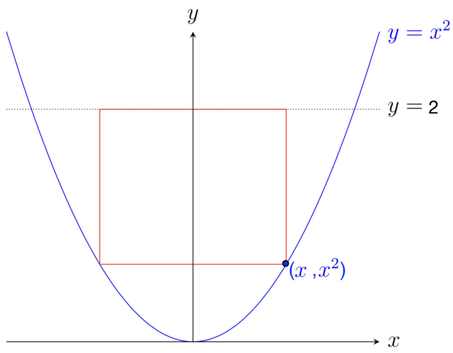
(d) (4pts) Follow through. -2 for each mistake.

(e) (2pts) 1 point for horizontal asymptote and 1 point for vertical asymptote.

(f) (5pts) Follow through. Their picture need to match their answers above. -1 for each item not labeled or different from previous answers.

7. (12 pts) 在拋物線 $y = x^2$ 內側且水平線 $y = 2$ 下方畫一長方形使得長方形的頂邊與水平線 $y = 2$ 重合 (見圖)， 求此長方形最大面積。

Find the largest rectangle that fits inside the graph of the parabola $y = x^2$ below the line $y = 2$, with the top side of the rectangle on the horizontal line $y = 2$.



Solution:

- 4M - for writing down the correct function to maximize
- 2M - correct derivative
- 2M - correct critical number
- 3M - any correct argument that verifies maximality of the critical number
- 1M - correct answer

Sample solution.
Let (x, x^2) be the coordinates of the right hand corner of the rectangle. We want to maximize the function

$$A(x) = (2x)(2 - x^2) = \underbrace{4x - 2x^3}_{4M}.$$

Differentiating gives

$$A'(x) = \underbrace{4 - 6x^2}_{2M}.$$

Setting $A'(x) = 0$ gives $x = \underbrace{\sqrt{\frac{2}{3}}}_{2M}$ (negative rejected).

(**) Since $A''\left(\sqrt{\frac{2}{3}}\right) = -12 \cdot \sqrt{\frac{2}{3}} < 0$,

the second derivative test implies that $A(x)$ attains a maximum at $x = \sqrt{\frac{2}{3}}$ ⋯ (3M)

and at which $A(\sqrt{\frac{2}{3}}) = \underbrace{\frac{8\sqrt{6}}{9}}_{1M}$.

Alternative for (**). (Using first derivative test)

x	0	⋯	$\sqrt{\frac{2}{3}}$	⋯	$\sqrt{2}$
$A'(x)$	+	+	0	−	−

Therefore, the first derivative test implies that $A(x)$ attains a maximum at $x = \sqrt{\frac{2}{3}}$.

and at which $A(\sqrt{\frac{2}{3}}) = \frac{8\sqrt{6}}{9}$.

Another alternative for (**). (Using Extreme Value Theorem)
Compare the critical value and values at boundaries :

$A(\sqrt{\frac{2}{3}}) = \underbrace{\frac{8\sqrt{6}}{9}}$ and $A(0) = A(\sqrt{2}) = 0$.

We conclude that $A(x)$ attains a maximum at $x = \sqrt{\frac{2}{3}}$.