1101模組13-16班 微積分1 期考解答和評分標準

1. Find the following limits.

(a) (5%) $\lim_{x\to\infty} \dfrac{-x^{\frac{3}{2}}+2x}{3x^{\frac{3}{2}}+3x-5}$ (b) (6%) $\lim_{x\to 0^+}\left(\dfrac{\tan 2x}{x}+\dfrac{1}{\ln x}\right)$ (c) (6%) $\lim_{x\to 0}(1+3x)^{\frac{1}{\arcsin 2x}}$

(d) (5%) $\lim_{x\to 0}\left(\dfrac{1}{2x}-\dfrac{1}{1-e^{-2x}}\right)$

Solution:

(a) $\lim_{x\to\infty} \dfrac{-x^{\frac{3}{2}}+2x}{3x^{\frac{3}{2}}+3x-5} = \lim_{x\to\infty}\dfrac{-1+2x^{-\frac{1}{2}}}{3+3x^{-\frac{1}{2}}-5x^{-\frac{3}{2}}}$ (3 pts) $= \dfrac{-1+2\cdot 0}{3+3\cdot 0-5\cdot 0} = -\dfrac{1}{3}$ (2 pts)

(b) $\lim_{x\to 0^+}\left(\dfrac{1}{\ln x}\right) = 0$ ($\because \lim_{x\to 0^+}\ln x = -\infty$) (1 pt)

$$\lim_{x\to 0^+}\frac{\tan 2x}{x} \overset{\text{L'H}}{\underset{\frac{0}{0}}{=\!=}} \lim_{x\to 0^+}\frac{2\sec^2 2x}{1} \text{(2 pts for } (\tan x)' = \sec^2 x.\text{ 1 pt for the chain rule.)}$$
$$= 2 \text{(1 pt)}$$

Hence $\lim_{x\to 0^+}\left(\dfrac{\tan 2x}{x}+\dfrac{1}{\ln x}\right) = \lim_{x\to 0^+}\left(\dfrac{\tan 2x}{x}\right) + \lim_{x\to 0^+}\dfrac{1}{\ln x} = 2$

Another solution for computing $\lim_{x\to 0^+}\dfrac{\tan 2x}{x}$.

$$\begin{aligned}\lim_{x\to 0^+}\frac{\tan 2x}{x} &= \lim_{x\to 0^+}\frac{\sin 2x}{x}\cdot\frac{1}{\cos 2x}\\ &= \lim_{x\to 0^+}\frac{\sin 2x}{2x}\cdot 2\cdot\frac{1}{\cos 2x} \quad \text{(2 pts)}\\ &= \left(\lim_{x\to 0^+}\frac{\sin 2x}{2x}\right)\cdot\left(\lim_{x\to 0^+} 2\cdot\frac{1}{\cos 2x}\right)\\ &= 1\cdot 2 = 2 \quad \text{(2 pts)}\end{aligned}$$

(Students can use the limit $\lim_{x\to 0^+}\dfrac{\sin 2x}{x} = 2$)

(c) $\ln\left((1+3x)^{\frac{1}{\arcsin 2x}}\right) = \dfrac{\ln(1+3x)}{\arcsin(2x)}$ (1 pt)

$$\lim_{x\to 0^+}\frac{\ln(1+3x)}{\arcsin(2x)} \overset{\text{L'H}}{\underset{\frac{0}{0}}{=\!=}} \lim_{x\to 0^+}\frac{\frac{3}{1+3x}}{\frac{2}{\sqrt{1-4x^2}}} \text{(1 pt for } (\ln(1+3x))'\text{, 2 pts for } (\arcsin(2x))'\text{)}$$
$$= \frac{3}{2} \quad \text{(1 pt)}$$

Hence $\lim_{x\to 0}(1+3x)^{\frac{1}{\arcsin 2x}} = e^{\frac{3}{2}}$ (1 pt)

(d)

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{2x}-\frac{1}{1-e^{-2x}}\right) &= \lim_{x\to 0}\frac{1-e^{-2x}-2x}{2x((1-e^{-2x})}\\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=\!=}} \frac{2e^{-2x}-2}{2(1-e^{-2x})+4xe^{-2x}} \quad \text{(2 pts)}\\ &= \lim_{x\to 0}\frac{e^{-2x}-1}{1-e^{-2x}+2xe^{-2x}} \overset{\text{L'H}}{\underset{\frac{0}{0}}{=\!=}} \lim_{x\to 0}\frac{-2e^{-2x}}{4e^{-2x}-4xe^{-2x}} \quad \text{(2 pts)}\\ &= -\frac{1}{2}. \quad \text{(1 pt)}\end{aligned}$$

2. Let $f(x) = \begin{cases} |x|\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

(a) (5%) Determine whether $f(x)$ is continuous at $x = 0$. Explain your answer.

(b) (5%) Determine whether $f(x)$ is differentiable at $x = 0$. Explain your answer.

Solution:

(a) Since $-1 \leq \cos(\frac{1}{x}) \leq 1$ for all $x \neq 0$, $-|x| \leq |x|\cos(\frac{1}{x}) \leq |x|$ for all $x \neq 0$.(2 points)

By Squeeze Theorem, $\lim_{x\to 0} |x|\cos(\frac{1}{x}) = 0$.(2 points)

Since $\lim_{x\to 0} |x|\cos(\frac{1}{x}) = f(0)$, $f(x)$ is continuous at $x = 0$.(1 points)

(b)

$$\begin{aligned} & \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0})\text{(1 points)} \\ = & \lim_{x\to 0^+} \frac{|x|\cos(\frac{1}{x})}{x})\text{(1 points)} \\ = & \lim_{x\to 0^+} \cos(\frac{1}{x})\text{(1 points)} \end{aligned}$$

Since $\lim_{x\to 0^+} \cos(\frac{1}{x})$ does not exist(1 points), Hence $f(x)$ is not differentiable at $x = 0$ (1 points).

3. Find $f'(x)$.

(a) (7%) $f(x) = \arctan\left(\frac{x}{2}\right) + \ln\sqrt{\frac{x-2}{x+2}}$, for $x > 2$.

(b) (8%) $f(x) = x^{\ln(x^3)} + 2^x$, for $x > 0$.

Solution:

(a)

$$\begin{aligned} \frac{d}{dx}\left(\arctan(\frac{x}{2})\right) &= \frac{1}{1+(x/2)^2}\cdot\frac{d}{dx}\left(\frac{x}{2}\right) \text{(2 points)} \\ &= \frac{2}{4+x^2} \text{(1 points)} \\ \frac{d}{dx}\left(\ln\sqrt{\frac{x-2}{x+2}}\right) &= \frac{1}{\sqrt{\frac{x-2}{x+2}}}\cdot\frac{d}{dx}\left(\sqrt{\frac{x-2}{x+2}}\right) \text{(1 points)} \\ &= \frac{1}{\sqrt{\frac{x-2}{x+2}}}\frac{1}{\sqrt{\frac{x-2}{x+2}}}\cdot\frac{2}{(x+2)^2} \text{(1 points)} \\ &= \frac{2}{x^2-4} \text{(1 points)} \end{aligned}$$

Hence $\frac{dy}{dx} = \frac{4x^2}{x^4-16}$ (1 points).

(b) Let $a = x^{\ln(x^3)}$. Thus

$$\begin{aligned} & \ln a = \ln(x^3)\ln x = 3(\ln x)^2 \text{(2 points)} \\ \Rightarrow\ & \frac{1}{a}\frac{da}{dx} = 6\ln x\cdot\frac{1}{x} \text{(2 points)} \\ \Rightarrow\ & \frac{da}{dx} = \frac{6x^{\ln(x^3)}\ln x}{x} \text{(1 points)} \end{aligned}$$

$$\frac{d}{dx}(2^x) = \ln 2\cdot 2^x \text{(2 points)}$$

Thus $\frac{dy}{dx} = \frac{6x^{\ln(x^3)}\ln x}{x} + \ln 2\cdot 2^x$ (1 points).

4. $\tan(x-y) = x^2 + \sin(2y)$ defines y as an implicit function of x near $(0,0)$ which is denoted by $y = y(x)$.

(a) (7%) Compute $\dfrac{dy}{dx}$ at $(0,0)$.

(b) (5%) Write down the linearization of $y(x)$ at $x = 0$. Use the linear approximation to estimate $y(0.01)$.

Solution:

(a) $\tan(x - y(x)) = x^2 + \sin(2y(x)) \xrightarrow{\frac{d}{dx}} \sec^2(x-y)\cdot(1-y') = 2x + 2\cos(2y)\cdot y'$
(4 pts total. 1 pt for $(\tan x)' = \sec^2 x$. 1 pt for $(\sin x)' = \cos x$. 2 pts for the chain rule.)
At $(x,y) = (0,0)$, $\sec^2(0)(1-y'(0)) = 0 + 2\cdot\cos 0\cdot y'(0)$
Hence $y'(0) = \dfrac{1}{3}$.
(3 pts total. 2 pt for plugging in $(x,y) = (0,0)$. 1 pts for solving $y'(0)$.)

(b) The linearization of $y(x)$ at $x = 0$ is

$$\begin{aligned} L(x) &= y(0) + y'(0)(x-0) && \text{(1 pt)} \\ &= 0 + \frac{1}{3}x = \frac{1}{3}x && \text{(2 pts)} \leftarrow \text{(a)算錯的話，這裡扣 1 分} \end{aligned}$$

Hence we can approximate $y(0.01)$ by $L(0.01)$ which is $L(0.01) = \dfrac{0.01}{3}$. (2 pts) ((a)算錯的話，這裡再扣 1 分)

5. *Dominant 7* is an acappella (阿卡貝拉) group formed by seven students at NTU. They have performed and won several international competitions. Currently they are planning to organise their first concert. When x tickets are demanded for their concert, the price of the tickets can be described by the function

$$p(x) = \frac{96}{\sqrt{x}} - \frac{x}{9} \text{ where } 1 \leq x \leq 90.$$

The cost in organising the concert is given by the function $C(x) = 0.001x^3 - 0.105x^2 + 300$.

(a) (6%) Find the maximum revenue generated by the ticket sales. (Revenue = Price × Quantity demanded)

(b) (5%) Discuss when the economy of scale occurs. i.e. the marginal cost $C'(x)$ is decreasing.

(c) (3%) Write down the profit function $\Pi(x)$. (Profit = Revenue − Cost).

(d) (4%) Student A claims that when the revenue is maximized, the profit is also maximized. Do you agree with Student A? Explain.

Solution:

(a)

Marking Scheme.
(1M) Writing down the correct revenue function
(2M) Find $R'(x)$ correctly (1M for each term)
(1M) Find the correct critical number $x = 36$
(2M) Any argument that $x = 36$ indeed gives a maximum value

Sample solution.

The revenue function is $R(x) = x \cdot p(x) = \underbrace{96\sqrt{x} - \frac{x^2}{9}}_{(1M)}$.

Differentiate : $\underbrace{R'(x) = \frac{48}{\sqrt{x}} - \frac{2x}{9}}_{(2M)}$. Set $R'(x) = 0$, we obtain $\underbrace{x = 36}_{(1M)}$.

(**) Since $R''(x) = -24x^{-\frac{3}{2}} - \frac{2}{9}$ and hence $R''(36) < 0$, the second derivative test implies that $R(x)$ attains a maximum at $x = 36$.

Alternative for (**). (Using first derivative test)

x	$\cdots$	36	$\cdots$
$R'(x)$	+	0	−

Therefore, the first derivative test implies that $R(x)$ attains a maximum at $x = 36$.

Another alternative for (**). (Using Extreme Value Theorem)

Compare the critical value and values at boundaries : $R(1) = 96 - \frac{1}{9}$, $R(36) = 96 \cdot 6 - 144$ and $R(90) = 96\sqrt{90} - 900$. We conclude that $R(x)$ attains a maximum at $x = 36$.

(b)

Marking Scheme.
(1M) Writing down $C''(x) < 0$ or $C''(x) \leq 0$
(2M) Find $C''(x)$ correctly
(2M) Solving the inequality correctly.
Remark 1. Accept both $<$ and $\leq$.
Remark 2. No deductions for omitting $1 \leq x$ in the final answer.

Sample solution.

The economy occurs when

$$\underbrace{C''(x) < 0}_{1M} \Leftrightarrow \underbrace{0.006x - 0.21}_{2M} < 0 \Leftrightarrow \underbrace{x < 35}_{2M}.$$

Hence, the economy occurs when $1 \leq x < 35$.

(c)

Marking Scheme.
Almost all or nothing. Minor deduction for obvious typo.

Sample solution.
$\Pi(x) = x\left(\frac{96}{\sqrt{x}} - \frac{x}{9}\right) - (0.0001x^3 - 0.105x^2 + 300) \cdots$ (3M)

(d)

Marking Scheme.
The key argument is to verify that $x = 36$ is not a critical number for the profit/cost function.
(2M) Correct approach
(2M) Verifying $\Pi'(36) = C'(36) \neq 0$
Remark. Just disagreeing student A or writing 'No' only without any elaborations receive no credits.
Remark. A student who gets (a) or (c) incorrect can get at most 2M here.

Sample solution 1. (Arguing by Profit)
If $\Pi(x)$ attains a maximum at $x = x_0$, Fermat's Theorem implies that $\Pi'(x_0) = 0$. $\cdots$ (2M)
However, $\Pi'(36) = R'(36) + C'(36) = 0 + C'(36) < 0.\cdots$ (2M)
Therefore, $\Pi(x)$ does not attain a maximum at $x = 36$. This disproves Student A's claim.

Sample solution 2. (Arguing by Cost)
If $\Pi(x)$ attains a maximum at $x = x_0$, then marginal cost and marginal revenue are equal.$\cdots$ (2M)
i.e. $R'(x_0) = C'(x_0)$.
However, $R'(36) = 0$ by (a) whereas $C'(36) < 0.\cdots$ (2M)
Therefore, $\Pi(x)$ does not attain a maximum at $x = 36$. This disproves Student A's claim.

6. $f(x) = (x^2 + x)^{\frac{1}{2}}$.

(a) (10%) Write down the domain of $f(x)$. Find all asymptotes of $y = f(x)$.

(b) (5%) Compute $f'(x)$. Find interval(s) of increase and interval(s) of decrease of $f(x)$.

(c) (5%) Compute $f''(x)$. Determine concavity of $y = f(x)$.

(d) (3%) Sketch the curve $y = f(x)$.

Solution:

(a)

Marking Scheme. Domain : 3 % + Slant asymptote : 3.5 % each
Domain :

- 1% - Solving $x(x+1) = 0 \Rightarrow x = 0$ or $x = -1$
- 2% - The correct domain

Asymptotes : (at $\pm\infty$ - 3.5% each)

- 1% - for computing $a = \lim_{x\to\pm\infty} \frac{f(x)}{x}$ correctly
- 1.5% - for computing $\lim_{x\to\pm\infty}(f(x) - ax)$ correctly
- 1% - for writing down the equation of the asymptote

Remark. Also okay if a student manages to guess an asymptote $y = ax + b$ and check $\lim_{x\to\pm\infty}(f(x) - ax - b) = 0$ directly.

Sample Solution.
(Domain)
The domain of $f = \{x|x^2 + x \geq 0\}$.
Let $h(x) = x^2 + x = x(x+1)$. Now we want to determine the sign graph of $h(x)$ We first solve $x^2 + x = 0$, i.e. $x(x+1) = 0 \Rightarrow x = 0$ or $x = -1$. (1 point)
These two points divide the real line into three subintervals $(-\infty,-1) \cup (-1,0) \cup (0,\infty)$. Evaluate $g(-2) = (-2)(-2+1) > 0$, $g(-0.5) = (-0.5)(-0.5+1) < 0$ and $g(1) = 1(1+1) > 0$. So $g(x) = x^2 + x > 0$ on $(-\infty,-1) \cup (0,\infty)$.
So the domain of f is $(-\infty,-1] \cup [0,\infty)$. (2 points)

Note that g is continuous on $(-\infty,-1] \cup [0,\infty)$. There is no vertical asymptote.

To find slant asymptotes, we first compute

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{(x^2+x)^{\frac{1}{2}}}{x} = \lim_{x\to\infty} \frac{(x^2(1+\frac{1}{x}))^{\frac{1}{2}}}{x}$$
$$= \lim_{x\to\infty} \frac{x((1+\frac{1}{x}))^{\frac{1}{2}}}{x} = \lim_{x\to\infty}((1+\frac{1}{x}))^{\frac{1}{2}} = 1.$$

(1 point)

Then we compute

$$
\begin{aligned}
&\lim_{x\to\infty}(x^2+x)^{\frac{1}{2}}-x\\
&=\lim_{x\to\infty}x(1+\frac{1}{x})^{\frac{1}{2}}-x=\lim_{x\to\infty}x[(1+\frac{1}{x})^{\frac{1}{2}}-1]\\
&=\lim_{x\to\infty}\frac{[(1+\frac{1}{x})^{\frac{1}{2}}-1]}{\frac{1}{x}}=\lim_{h\to0^+}\frac{[(1+h)^{\frac{1}{2}}-1]}{h}\\
&=\lim_{h\to0^+}\frac{[(1+h)^{\frac{1}{2}}-1]'}{h'}=\lim_{h\to0^+}\frac{\frac{1}{2}(1+h)^{-\frac{1}{2}}}{1}\\
&=\frac{1}{2}.
\end{aligned}
$$

(1.5 point)

Thus $\lim_{x\to\infty}(x^2+x)^{\frac{1}{2}}-(x+\frac{1}{2})=0$ or equivalently $y=x+\frac{1}{2}$ is a slant asymptote. (1 point).

On the other hand,

$$
\begin{aligned}
\lim_{x\to-\infty}\frac{f(x)}{x}&=\lim_{x\to-\infty}\frac{(x^2+x)^{\frac{1}{2}}}{x}=\lim_{x\to-\infty}\frac{(x^2(1+\frac{1}{x}))^{\frac{1}{2}}}{x}\\
&=\lim_{x\to-\infty}\frac{-x((1+\frac{1}{x}))^{\frac{1}{2}}}{x}=\lim_{x\to\infty}-((1+\frac{1}{x}))^{\frac{1}{2}}=-1.
\end{aligned}
$$

(1 point)

Now we compute

$$
\begin{aligned}
&\lim_{x\to-\infty}(x^2+x)^{\frac{1}{2}}+x\\
&=\lim_{x\to-\infty}-x(1+\frac{1}{x})^{\frac{1}{2}}+x=\lim_{x\to-\infty}x[-(1+\frac{1}{x})^{\frac{1}{2}}+1]\\
&=\lim_{x\to-\infty}\frac{[-(1+\frac{1}{x})^{\frac{1}{2}}+1]}{\frac{1}{x}}=\lim_{h\to0^-}\frac{[-(1+h)^{\frac{1}{2}}+1]}{h}\\
&=\lim_{h\to0^-}\frac{[-(1+h)^{\frac{1}{2}}+1]'}{h'}=\lim_{h\to0^-}\frac{-\frac{1}{2}(1+h)^{-\frac{1}{2}}}{1}\\
&=-\frac{1}{2}
\end{aligned}
$$

(1.5 point)

Thus $\lim_{x\to-\infty}(x^2+x)^{\frac{1}{2}}-(-x-\frac{1}{2})=0$ and $y=-x-\frac{1}{2}$ is also also slant asymptote. (1 point).

(b)

Marking Scheme.
2% - correct $f'(x)$
1% - for correctly determining the signs of each sub-intervals
1% - for the correct interval of increase
1% - for the correct interval of decrease

Sample solution.

$f'(x)=\frac{d}{dx}(x^2+x)^{\frac{1}{2}}=\frac{1}{2}(x^2+x)^{-\frac{1}{2}}(2x+1)$. (2 point)

Recall the domain of f is $(-\infty,-1]\cup[0,\infty)$. $f'(x)=0$ has no solution in its domain. $f'(-2)=\frac{1}{2}(4-2)^{-\frac{1}{2}}(-4+1)<0$ and $f'(1)=\frac{1}{2}(1+1)^{-\frac{1}{2}}(2+1)>0$.

. (1 point for signs)

So $f'(x)<0$ and f is decreasing on $(-\infty,-1)$. (1 point)

$f'(x)>0$ and f is increasing on $(0,\infty)$. (1 point)

(c)

Marking Scheme.
3% - correct $f''(x)$
2% - for the correct interval/explaining why the curve is always concaving downward

Sample Solution.
Using $f'(x) = \frac{1}{2}(x^2+x)^{-\frac{1}{2}}(2x+1)$, we have

$$\begin{aligned} f''(x) =& \frac{1}{2}([(x^2+x)^{-\frac{1}{2}}]'(2x+1) + (x^2+x)^{-\frac{1}{2}}(2x+1)') \\ =& \frac{1}{2}(-\frac{1}{2}(x^2+x)^{-\frac{3}{2}}(2x+1)^2 + (x^2+x)^{-\frac{1}{2}}2) \\ =& -\frac{1}{4}(x^2+x)^{-\frac{3}{2}}((2x+1)^2 - 4(x^2+x)) \\ =& -\frac{1}{4}(x^2+x)^{-\frac{3}{2}}(4x^2+4x+1-4(x^2+x)) \\ =& -\frac{1}{4}(x^2+x)^{-\frac{3}{2}}. \end{aligned}$$

(3 point)

Recall the domain of f is $(-\infty,-1] \cup [0,\infty)$ and $x^2+x > 0$ on $(-\infty,-1) \cup (0,\infty)$. So $f''(x) < 0$ and f is concave down on $(-\infty,-1) \cup (0,\infty)$. (2 points)

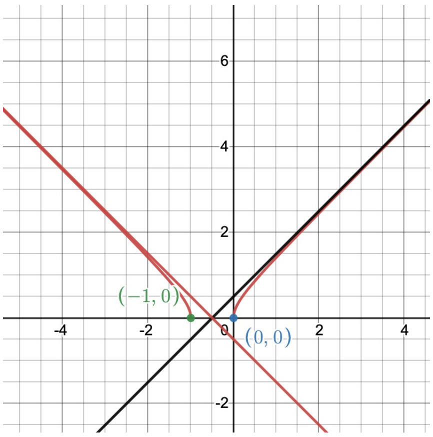
(d)

Marking Scheme.
1% - appropriate 'ends' (or x-intercepts or one-sided vertical tangents)
1% - the two slant asymptotes
1% - the shape (increase, decrease, always concaving downward)

Remark. It's okay that the students do not demonstrate vertical-ness of the one-sided tangents at the points $x = 0$ and $x = -1$.

Sample solution.
Note that $f(-1) = f(0) = 0$. (1 point)
f is decreasing and concave down on $(-\infty,-1)$. f is increasing and concave down on $(0,\infty)$.

Also, note that $\lim_{x\to-1^-} f'(x) = \lim_{x\to-1^-} \frac{1}{2}(x^2+x)^{-\frac{1}{2}}(2x+1) = -\infty$ and. $\lim_{x\to0^+} f'(x) = \lim_{x\to0^+} \frac{1}{2}(x^2+x)^{-\frac{1}{2}}(2x+1) = \infty$.
f has a one sided vertical tangent at $x = -1$ and $x = 0$



(Slant asymptotes - 1 point) (Correct shape - 1 point)