1. (12 pts) Compute the following limits. If they don't exist, find the one-sided limits.

(a) (6 pts) $\lim_{x\to 0} \frac{\sqrt{1-\cos 2x}}{x}$ (b) (6 pts) $\lim_{x\to 0} \frac{x^2\cos\left(\frac{1}{x}\right)}{\tan x}$

Solution:

(a) $\lim_{x\to 0} \frac{\sqrt{1-\cos 2x}}{x} = \lim_{x\to 0} \frac{\sqrt{2\sin^2 x}}{x} = \lim_{x\to 0} \frac{\sqrt{2}|\sin x|}{x}$ (2%)

For $x > 0$, we have $|\sin x| = \sin x$. Then we have

$$\lim_{x\to 0^+} \frac{\sqrt{1-\cos 2x}}{x} = \lim_{x\to 0^+} \frac{\sqrt{2}|\sin x|}{x} = \lim_{x\to 0^+} \frac{\sqrt{2}\sin x}{x} = \sqrt{2}. \quad (1\%)$$

For $x < 0$, we have $|\sin x| = -\sin x$. Then we have

$$\lim_{x\to 0^-} \frac{\sqrt{1-\cos 2x}}{x} = \lim_{x\to 0^-} \frac{\sqrt{2}|\sin x|}{x} = \lim_{x\to 0^-} \frac{-\sqrt{2}\sin x}{x} = -\sqrt{2}. \quad (1\%)$$

Since $\lim_{x\to 0^+} \frac{\sqrt{1-\cos 2x}}{x} \neq \lim_{x\to 0^-} \frac{\sqrt{1-\cos 2x}}{x}$, we have $\lim_{x\to 0} \frac{\sqrt{1-\cos 2x}}{x}$ does not exist. (2%)

(b)

$$\lim_{x\to 0} \frac{x^2\cos(\frac{1}{x})}{\tan x} = \lim_{x\to 0} \cos x \cdot \frac{x}{\sin x} \cdot [x\cos(\frac{1}{x})]$$

Since $-1 \le \cos(\frac{1}{x}) \le 1$ for $x \neq 0$, we have $-|x| \le x\cos(\frac{1}{x}) \le |x|$ for $x \neq 0$ (2%). By $\lim_{x\to 0} -|x| = \lim_{x\to 0}|x| = 0$ and squeeze theorem, we have $\lim_{x\to 0} x\cos(\frac{1}{x}) = 0$ (2%). Therefore, by $\lim_{x\to 0} \frac{x}{\sin x} = 1$, we have

$$\begin{aligned}\lim_{x\to 0} \frac{x^2\cos(\frac{1}{x})}{\tan x} &= \lim_{x\to 0} \cos x \cdot \frac{x}{\sin x} \cdot [x\cos(\frac{1}{x})] \\ &= \lim_{x\to 0} \cos x \cdot \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} [x\cos(\frac{1}{x})] = 1 \cdot 1 \cdot 0 = 0 \ (2\%).\end{aligned}$$

2. (14 pts) Compute the given derivative.

(a) (7 pts) Given $e^{xy}\ln\frac{x}{y} = x + \frac{1}{y}$, find $\frac{dy}{dx}$ at $\left(e, \frac{1}{e}\right)$.

(b) (7 pts) $y = \left(\tan^{-1} x\right)^{\sec(2x-2)}$, find $\frac{dy}{dx}$ at $x = 1$.

Solution:

(a)

$$e^{xy(x)}(\ln x - \ln(y(x))) = x + \frac{1}{y(x)}$$
$$\xrightarrow{\frac{d}{dx}} e^{xy}(y + xy')(\ln x - \ln y) + e^{xy}(\frac{1}{x} - \frac{1}{y}y') = 1 - \frac{1}{y^2}y' ......(*)$$

(4 pts total. 1 pt for product rule. 1 pt for $(e^x)'$. 1 pt for $(\ln x)'$. 1 pt for the chain rule.)

At $(x, y) = (e, \frac{1}{e})$, the equation (*) becomes

$$e^1(\frac{1}{e} + ey')(1+1) + e^1(\frac{1}{e} - ey') = 1 - e^2y'$$
$$\Rightarrow 2(1 + e^2y') + 1 - e^2y' = 1 - e^2y'$$
$$\Rightarrow y' = -\frac{1}{e^2}.$$

(3 pts. 2 pts for plugging in $(x, y) = (e, \frac{1}{e})$ correctly. 1 pt for sloving y'.)

(b)

$$\ln y(x) = \sec(2x-2)\ln(\tan^{-1} x) \quad \text{(1 pt)}$$
$$\xrightarrow{\frac{d}{dx}} \frac{y'(x)}{y(x)} = \sec(2x-2)\tan(2x-2)\cdot 2\cdot\ln(\tan^{-1} x) + \sec(2x-2)\frac{1}{\tan^{-1} x}\frac{1}{1+x^2}$$

(4 pts totals. 1 pt for $(\sec x)'$. 1 pt for $(\ln x)'$, 1 pt for $(\tan^{-1} x)'$. 1 pt for the chain rule)

At $x = 1$, $y(1) = (\tan^{-1} 1)^{\sec 0} = (\frac{\pi}{4})' = \frac{\pi}{4}$ (1 pt)

$y'(1) = \frac{1}{2}$ (1 pt)

3. (14 pts) Let $f(x) = x^{-2x}$, $x > 0$.

(a) (6 pts) Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to\infty} f(x)$.

(b) (8 pts) Use the linearization of $f(x)$ at $x = 1$ to estimate $0.97^{-1.94}$.

Solution:

The following crucial steps must be shown clearly

(a) a.1) (1%) $\lim_{x\to 0^+} f(x) = e^{\lim_{x\to 0^+} -2x\,\ln(x)}$

(2%) $= e^{-2\lim_{x\to 0^+} \frac{\ln(x)}{1/x}} = e^{-2\lim_{x\to 0^+} \frac{1/x}{-1/x^2}}$

(1%) $= e^{2\lim_{x\to 0^+} x} = e^0 = 1$

a.2) (2%) $\lim_{x\to\infty} f(x) = e^{\lim_{x\to\infty} -2x\,\ln(x)} = e^{-\infty} = 0$

(b) (2%) $f(1) = e^{-2\ln(1)} = e^0 = 1$

(3%) $f'(x) = -2(1 + \ln x)f(x) \Rightarrow f'(1) = -2$

(3%) $f(0.97) \doteq f(1) + f'(1)(0.97 - 1) = 1 - 2(-0.03) = 1.06$

4. (10 pts) Consider the function

$$f(x) = \begin{cases} A \cdot \dfrac{\sin x}{x} & \text{if } x < 0 \\ B \cdot \cos\left(\sqrt{x}\right) + e^{3x} & \text{if } x \geq 0 \end{cases}$$

where A and B are constants. If f is differentiable everywhere, find the values of A and B.

Solution:

Marking Scheme.
1M - definition of continuity
1M - obtaining the equation $A = B + 1$ by continuity
2M - (**) definition of differentiability
3M - (***) correct evaluation of $\lim\limits_{h \to 0^+} \dfrac{f(h) - f(0)}{h} = -\dfrac{B}{2} + 3$
2M - (****) correct evaluation of $\lim\limits_{h \to 0^-} \dfrac{f(h) - f(0)}{h} = 0$
1M - correct answer

Remark.
(i) If a student computed the answer by setting $\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^-} f'(x)$ and clearly misinterpreted this as the definition of differentiability (or failed to make correct connection of this equality with the definition of differentiability in this case), the 2M from (**) will be taken away - so he/she will earn at most 8M from this question.
(ii) Partial credits are possible to (***) or (****) for minor mistakes (sign errors, clear typos). Correct use of L'Hospital's rule is allowed.

Sample Solution.
Given that f is differentiable at $x = 0$. This implies f is continuous at $x = 0$. Therefore, we have

$$\begin{aligned} &\lim_{x \to 0^-} f(x) = f(0) \cdots\cdots (1\text{M}) \\ \Rightarrow &\lim_{x \to 0^-} A \cdot \frac{\sin x}{x} = B \cos 0 + e^0 \\ \Rightarrow &A = B + 1 \cdots\cdots\cdots\cdots (1\text{M}) \end{aligned}$$

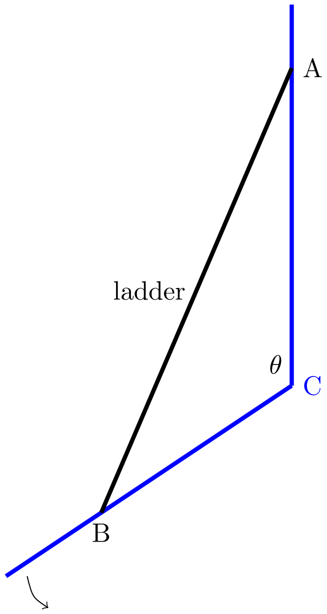
Moreover, the differentiability of f at $x = 0$ implies that the limits

$$\underbrace{\lim_{h \to 0^+} \frac{f(h) - f(0)}{h}}_{(2\text{M})} = \lim_{h \to 0^+} \frac{(B\cos(\sqrt{h}) + e^{3h}) - (B+1)}{h} = \lim_{h \to 0^+} B \cdot \frac{\cos(\sqrt{h}) - 1}{h} + \frac{e^{3h} - 1}{h} = \underbrace{-\frac{B}{2} + 3}_{(3\text{M})}$$

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{A \cdot \frac{\sin h}{h} - (B+1)}{h} &= \lim_{h \to 0^-} (B+1) \cdot \frac{\frac{\sin h}{h} - 1}{h} \\ &= (B+1) \cdot \lim_{h \to 0^-} \frac{\sin h - h}{h^2} \\ &\overset{\text{L'H}}{=} (B+1) \cdot \lim_{h \to 0^-} \frac{\cos h - 1}{2h} = \underbrace{0}_{(2\text{M})} \end{aligned}$$

are equal so we have $-\dfrac{B}{2} + 3 = 0 \Rightarrow B = 6$ and hence $A = 7$. $\cdots\cdots$ (1M)

5. (15 pts) A ladder of length $\overline{AB} = 10$ m slides down along a wall when the angle C is opening at a rate $\frac{d\theta}{dt} = 0.2$ rad/s. If the top of the ladder slides down at a rate of 2 m/s, how fast is the bottom of the ladder sliding when both $\overline{AC}$ and $\overline{BC}$ are $\frac{10}{\sqrt{3}}$ m? (Recall the law of cosine: $z^2 = x^2 + y^2 - 2xy\cos\theta$.)



Solution:

Let us denote $\overline{AC}$ by x and $\overline{BC}$ by y. According to the law of cosine, we have

$$100 = x^2 + y^2 - 2xy\cos\theta.$$ (2 points for setting up the equation.)

Since x, y, and θ are functions of time t, after differentiating with respect to t, we get

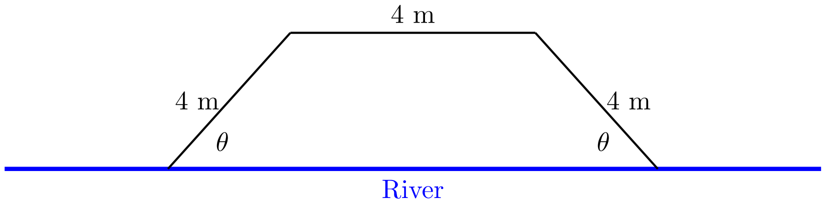
$$0 = 2xx' + 2yy' - 2x'y\cos\theta - 2xy'\cos\theta + 2xy\theta'\sin\theta. \quad (1)$$ (7 points for implicit differentiation.)

We already know that $\theta' = 0.2$ and $x' = -2$. The moment when $\overline{AC}$ and $\overline{BC}$ are both $10/\sqrt{3}$, the angle θ will satisfy $\cos\theta = -1/2$ which can be see by the law of cosine, so $\theta = 2\pi/3$. (3 points for finding θ.) Hence (1) becomes

$$0 = 2\frac{10}{\sqrt{3}}(-2) + 2\frac{10}{\sqrt{3}}y' - 2(-2)\frac{10}{\sqrt{3}}\frac{-1}{2} - 2\frac{10}{\sqrt{3}}y'\frac{-1}{2} + 2\frac{10}{\sqrt{3}}\frac{10}{\sqrt{3}}\frac{\sqrt{3}}{2}0.2. \quad (2)$$

Solving for y' to get $y' = 4/3$. (3 points for solving y'.)

6. (15 pts) A farmer has 12 m of fencing and wants to fence off a trapezoidal field that borders a straight river according to the picture below. He needs no fence along the river. What should the angle θ be in order to maximize the area of the enclosed field? What is the maximum area?



Solution:

We first observe that the problem makes sense for $0 \le \theta \le \dfrac{2\pi}{3}$. For any θ more than $2\pi/3$ the two slanted fence would intersect. But $\theta = \dfrac{\pi}{2}$ can be considered as the boundary as well.

We can find the area of the trapezoid by using the sum of the rectangular and the two triangular parts.

$$\text{Area} = 4 \cdot (4\sin\theta) + 2 \cdot \left(\frac{1}{2}(4\cos\theta)(4\sin\theta)\right) = 16\sin\theta + 16\sin\theta\cos\theta$$

Let $f(\theta)$ be the area function.

$$f(0) = 0, \quad f(\pi/2) = 16, \quad f(2\pi/3) = 8\sqrt{3} - 4\sqrt{3} = 4\sqrt{3},$$

$$f'(\theta) = 16(\cos\theta + \cos^2\theta - \sin^2\theta) = 16(2\cos^2\theta + \cos\theta - 1) = 16(2\cos\theta - 1)(\cos\theta + 1)$$

The function f is increasing on $[0, \pi/3]$ and decreasing on $[\pi/3, 2\pi/3]$. Absolute maximum occurs at $\theta = \pi/3$ with the value

$$f(\pi/3) = 8\sqrt{3} + 4\sqrt{3} = 12\sqrt{3}$$

□

For the second derivative test method,

$$f''(\theta) = -16(\sin\theta + 4\sin\theta\cos\theta), \quad f''(\pi/3) = -24\sqrt{3} < 0.$$

□

The maximum area is $12\sqrt{3}\ \text{m}^2$ and it is obtained at $\theta = \dfrac{\pi}{3}$.

Grading scheme: (15 points)

Optimization problems have a few steps, so we make sure to award follow through points.

Key steps:

(1) Area function (3 points): Since the variable is given, unless students introduce a new variable, otherwise this part is all or nothing. But even with a wrong function, they can continue to get points in other parts.

(2) Domain (2 points): This can show up anywhere and in many forms. They can earn the 2 points as long as they acknowledge the existence of bounds and check function values at the endpoints.

(3) Critical numbers (5 points): This includes finding the correct derivative, correctly solving the roots, and discuss which ones are relevant to our problem. -3 for any major mistake and -1 for minor mistakes.

(4) Justify maximum (3 points): They can either use extreme value theorem, first derivative test, second derivative test, or other methods. -2 if their explanation is too vague. (example: by looking at $f'(\theta)$ we see that the area is smaller for any other theta value.)

(5) Final answer (2 points): This is a word problem, so they need to state the answer properly. They need to at least label their answers and give them units. -1 for each mistake.

7. (20 pts) Let $a > 0$ be a positive constant and consider the function $g(x) = \dfrac{ax}{x^2+1}$.

(a) (5 pts) Find $g'(x)$. Write down the interval(s) of increase and interval(s) of decrease of $g(x)$.

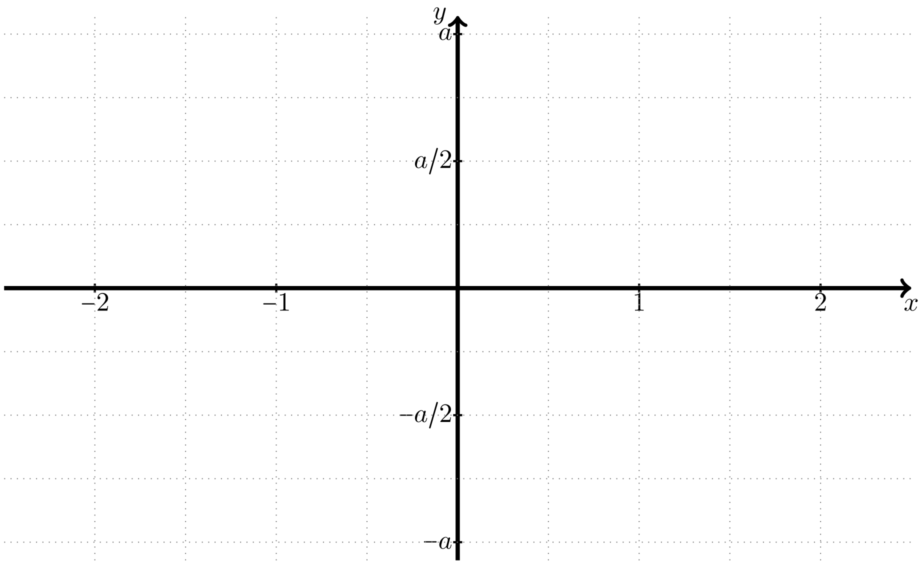
(b) (5 pts) Find $g''(x)$. Write down the interval(s) on which $g(x)$ is concave upward and the interval(s) on which $g(x)$ is concave downward.

(c) (4 pts) Write down (if any) the local extrema and inflection points.

(d) (3 pts) Sketch the graph of $y = g(x)$ and label all information you found above on the given grid below.

(e) (2 pts) Consider the function $f(x) = \tan(g(x)) = \tan\left(\dfrac{ax}{x^2+1}\right)$. Determine the range of a for which $f(x)$ is continuous on all real numbers $\mathbb{R}$.

(f) (1 pt) Determine the number of vertical asymptotes of the curve $y = f(x)$ if a is equal to 8.



Solution:

(a) By the quotient rule (or by any other method), $g'(x) = \dfrac{a(1-x^2)}{(x^2+1)^3}$ (2 pts if $g'(x)$ is calculated correctly (simplification is not necessary)). Thus, $g'(x) = 0 \Leftrightarrow a(1-x^2) = 0 \Leftrightarrow x = \pm 1$ (1 pt if the critical numbers $x = \pm 1$ are determined). If $|x| > 1$, $1 - x^2 < 0$ and if $|x| < 1$, $1 - x^2 > 0$. Thus,

$$\begin{aligned} g'(x) > 0\,(g : \text{increasing}) \Leftrightarrow &\quad x \in (-1,1) \quad (1\text{ pt}),\\ g'(x) < 0\,(g : \text{decreasing}) \Leftrightarrow &\quad x \in (-\infty,-1) \cup (1,\infty) \quad (1\text{ pt}). \end{aligned}$$

Other equivalent expressions such as expressing "$x \in (-1,1)$" as "$-1 < x < 1$" or "$|x| < 1$" are also fine. If the answer is given correctly but without a sufficient explanation, 4 points will be given.

(b) By the quotient rule (or by any other method), $g''(x) = \dfrac{2ax(x^2-3)}{(x^2+1)^3}$ (2 pts if $g'(x)$ is calculated correctly (simplification is not necessary)). Thus, $g''(x) = 0 \Leftrightarrow 2ax(x^2-3) = 0 \Leftrightarrow x = 0, \pm\sqrt{3}$ (1 pt if the critical numbers $x = 0, \pm\sqrt{3}$ of $g'(x)$ are determined). If $|x| > \sqrt{3}$, $x^2 - 3 > 0$ and if $|x| < \sqrt{3}$, $x^2 - 3 < 0$ and thus

$$\begin{aligned} g''(x) > 0\,(g : \text{concave up}) \Leftrightarrow &\quad x \in (-\sqrt{3},0) \cup (\sqrt{3},\infty) \quad (1\text{ pt}),\\ g''(x) < 0\,(g : \text{concave down}) \Leftrightarrow &\quad x \in (-\infty,-\sqrt{3}) \cup (0,\sqrt{3}) \quad (1\text{ pt}). \end{aligned}$$

If the student compute both of

$$\begin{aligned} g''(x) > 0 \Leftrightarrow &\quad x \in (-\sqrt{3},0) \cup (\sqrt{3},\infty) \quad (1\text{ pt}),\\ g''(x) < 0 \Leftrightarrow &\quad x \in (-\infty,-\sqrt{3}) \cup (0,\sqrt{3}) \quad (1\text{ pt}). \end{aligned}$$

correctly but switches (or does not mention) "concave up" and "concave down", 1 out of 2 points will be given.

(c) By (a), $x = -1$ is where $g'(x)$ changes its sign from negative to positive, and so

$$(x,y) = (-1, g(-1)) = \left(-1, -\frac{a}{2}\right)$$

is the unique local minimum of $y = g(x)$ (1 pts). Similarly, $x = 1$ is where $g'(x)$ changes its sign from positive to negative, and so

$$(x,y) = (1, g(1)) = \left(1, \frac{a}{2}\right)$$

is the unique local maximum of $y = g(x)$ (1 pts). By (2), $x = 0, \pm\sqrt{3}$ are where $g''(x)$ changes its sign, so

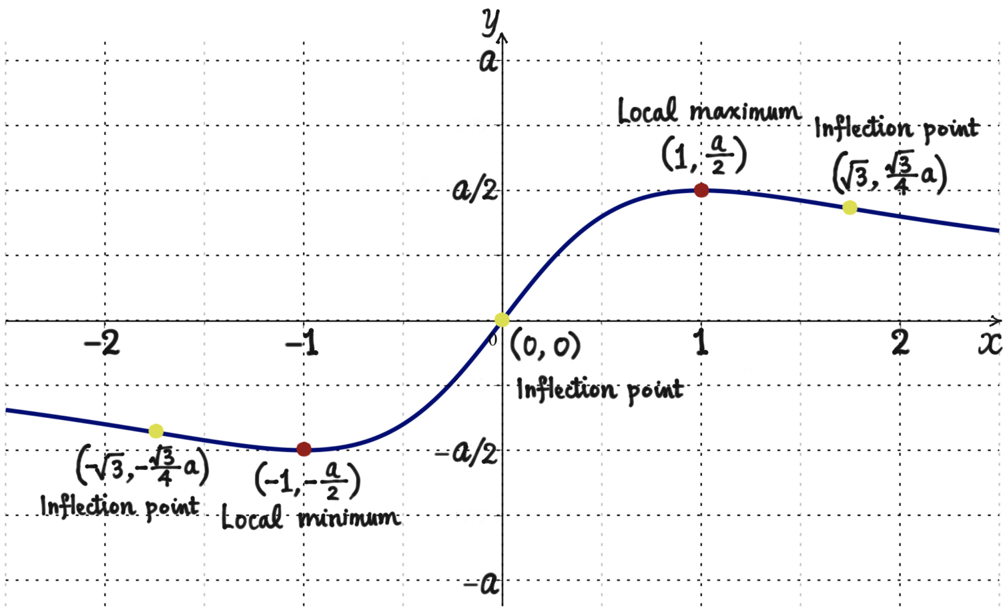
$$(x,y) = (0, g(0)) = (0,0)$$

and

$$(x,y) = (\pm\sqrt{3}, g(\pm\sqrt{3})) = \left(\pm\sqrt{3}, \frac{\pm\sqrt{3}}{4}a\right)$$

are the inflection points of $y = g(x)$ (2 pts if the students find all three inflection points, if one or two are missing, they will be given 1 pt). Even if the student only mentions the x-coordinates of the points, he/she still gets the full mark.

(d) 1 pt will be given if all the local extrema of $g(x)$ are indicated correctly, and the interval(s) of increase and the interval(s) of decrease of $g(x)$ can be seen. 1 pt will be given if all the inflection points are indicated correctly, and concave up and down are seen visually (no need to be too strict). 1 pt will be given if there is no fatal error (e.g. $g(x)$ intersecting the x-axis at $x \neq 0$).



(e) Since the smallest value of u where $\tan u$ is discontinuous is $u = \pm\frac{\pi}{2}$ and $\max|g(x)| = \frac{a}{2}$, $g(x) = \pm\frac{a}{2}$ do not occur if $a < \pi$. Therefore, $f(x) = \tan(g(x))$ is continuous on all real numbers $\mathbb{R}$ in this case. On the other hand, if $a \geq \pi$,

$$0 = g(0) < \frac{\pi}{2} \leq g(1) = \frac{a}{2},$$

hence there exists $0 < c \leq 1$ such that $g(c) = \frac{\pi}{2}$ by the intermediate value theorem. For marking, it is not necessary to check the complete logical details. For example, graphical understanding will be considered to be sufficient. If the student notices that the existence of the solution to $g(x) = \pm\frac{\pi}{2}$ gives rise to vertical asymptote, he/she will be given 1 point. If he/she notices that this occurs if and only if the local extrema $f(1) = \frac{a}{2}$ (or $f(1) = -\frac{a}{2}$) exceeds $\frac{\pi}{2}$ (or $-\frac{\pi}{2}$), he/she will be given another 1 pt (full mark).

(f) The vertical asymptotes of $\tan u$ occurs exactly when $u = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. Therefore, the number of vertical asymptotes of $\tan(g(x))$ is the total number of solutions for

$$g(x) = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

Since $a = 8$, $g(x) \le g(1) = \frac{a}{2} = 4$ and $g(x) \ge g(-1) = -\frac{a}{2} = -4$. Since $3 < \pi$, we have $g(x) \le 4 < \frac{3\cdot 3}{2} < \frac{3\pi}{2}$ and $g(x) \ge -4 > -\frac{3\pi}{2}$ and so $g(x) = \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ has no solutions. On the other hand,

$$0 = g(0) < \frac{\pi}{2} < g(1) = 4$$

shows that there is at least one solution to $g(x) = \frac{\pi}{2}$ on $x \in (0,1)$. Since $g(x)$ is strictly increasing on $(0,1)$, that is the only solution (since monotonicity $\Rightarrow$ one-to-one). Similarly, since

$$0 = \lim_{x\to\infty} g(x) < \frac{\pi}{2} < g(1) = 4$$

and $g(x)$ is strictly decreasing on $(1,\infty)$, $g(x) = \frac{\pi}{2}$ has a unique solution on $x \in (1,\infty)$. By the same argument (or by the symmetry $g(-x) = -g(x)$), $g(x) = -\frac{\pi}{2}$ has exactly one solution on $(-\infty,-1)$ and $(-1,0)$. Therefore, in total, $g(x) = \pm\frac{\pi}{2}$ has four solutions, which in turn means that $\tan(g(x))$ has exactly four vertical asymptotes.

When marking, the markers do not have to check the logical details. If the student seems to understand that the solutions to

$$g(x) = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

give rise to vertical asymptotes and thereby arrive at the correct answer, 1 pt is given.