1101模組01-05班 微積分1 期考解答和評分標準

1. (21 pts) Compute the limits or show that the limit doesn't exist.

(a) (7 pts) $\lim\limits_{x\to 1}\dfrac{2^x-2}{|x^2-x|}$ (b) (7 pts) $\lim\limits_{x\to 0}|\sin x|\cdot\sin\left(\dfrac{1}{x}\right)$ (c) (7 pts) $\lim\limits_{h\to 0}\dfrac{\ln(\cos(4h))}{h^2}$

Solution:

(a) $\lim\limits_{x\to 1^+}\dfrac{2^x-2}{|x^2-x|}=\lim\limits_{x\to 1^+}\dfrac{2^x-2}{x(x-1)}$ (1 pt)

Moreover, by the definition of $\dfrac{d}{dx}2^x\Big|_{x=1}$ or the L'Hospital's Rule,

$$\lim_{x\to 1^+}\frac{2^x-2}{x-1}=2\ln 2 \quad \text{(2 pts)}$$

Hence $\lim\limits_{x\to 1^+}\dfrac{2^x-2}{|x^2-x|}=\left(\lim\limits_{x\to 1^+}\dfrac{1}{x}\right)\cdot\left(\lim\limits_{x\to 1^+}\dfrac{2^x-2}{x-1}\right)=2\ln 2.$

Similarly, $\lim\limits_{x\to 1^-}\dfrac{2^x-2}{|x^2-x|}=\lim\limits_{x\to 1^-}\dfrac{2^x-2}{x(1-x)}=-2\ln 2$ (3 pts)

$\because \lim\limits_{x\to 1^+}\dfrac{2^x-2}{|x^2-x|}\neq\lim\limits_{x\to 1^-}\dfrac{2^x-2}{|x^2-x|}$ $\therefore \lim\limits_{x\to 1}\dfrac{2^x-2}{|x^2-x|}$ doesn't exist. (1 pt)

(b) $\because -1\le\sin\left(\dfrac{1}{x}\right)\le 1$ for all $x\neq 0$.

$\therefore -|\sin x|\le|\sin x|\sin\left(\dfrac{1}{x}\right)\le|\sin x|$ (3 pts)

Since $\lim\limits_{x\to 0}\sin x=0$, we have $\lim\limits_{x\to 0}|\sin x|=0$ and $\lim\limits_{x\to 0}-|\sin x|=0$ (2 pts)

Hence by the Squeeze theorem,

$$\lim_{x\to 0}|\sin x|\cdot\sin\left(\frac{1}{x}\right)=0 \quad \text{(2 pts)}$$

NOTE:

(1) 使用 $\lim\limits_{x\to 0}\dfrac{\sin\frac{1}{x}}{\frac{1}{x}}=1$ or product rule 震盪等等之類的說明，最多得1分。

(2) 使用 $\lim\limits_{x\to 0}\dfrac{\sin\frac{1}{x}}{\frac{1}{x}}=0$ 扣 2-4分 (主要扣的原因是沒使用到夾擠定理)

(c)

$$\lim_{h\to 0}\frac{\ln(\cos(4h))}{h^2}\overset{\text{L'H}}{\underset{\frac{0}{0}}{=\!=}}\lim_{h\to 0}\frac{\frac{-\sin(4h)}{\cos(4h)}\cdot 4}{2h}$$

$$\overset{\text{sol 1}}{=\!=\!=}\lim_{h\to 0}\frac{-8}{\cos(2h)}\frac{\sin(4h)}{4h}=-8\cdot 1=-8\text{(3 pts)}$$

$$\overset{\text{sol 2}}{=\!=\!=}\lim_{h\to 0}\frac{-2\sin(4h)}{h\cos(4h)}$$

$$\overset{\text{L'H}}{\underset{\frac{0}{0}}{=\!=}}\lim_{h\to 0}\frac{-8\cos(4h)}{\cos(4h)-4h\sin(4h)} \quad \text{(2 pts)}$$

$$=\frac{-8}{1}=-8 \quad \text{(1 pt)}$$

$$\overset{\text{sol 3}}{=\!=\!=}\lim_{h\to 0}\frac{-2\tan(4h)}{h}\overset{\text{L'H}}{\underset{\frac{0}{0}}{=\!=}}\lim_{h\to 0}\frac{-2\sec^2(4h)\cdot 4}{1}=-8$$

4 pts total. 1 pt for L'H. 1 pt for $(\ln x)'$. 1 pt for $(\cos x)'$. 1 pt for the chain rule constant.

2. (14 pts) Suppose that the equation

$$x^2\cos(xy) + e^{y^2} - 2x + y = 0$$

is satisfied by a differentiable function $y(x)$ defined on an open interval I containing 1 such that $y(1) = 0$. Besides, we assume that y'' exists everywhere on I.

(a) (6 pts) Compute $y'(1)$.

(b) (6 pts) Compute $y''(1)$.

(c) (2 pts) Does $y(x)$ attain a local extremum at $x = 1$? if your answer is YES, tell the type of local extremum (local maximum or local minimum) and give your reason.

Solution:

(a) Applying the implicit differentiation, we have

$$2x\cos(xy) - x^2\sin(xy)(y + xy') + e^{y^2}2yy' - 2 + y' = 0.$$

Setting $x = 1$ and $y(1) = 0$, we obtain that

$$2\cos(0) - 1^2 \cdot \sin(0)(0 + y'(1)) + e^0 \cdot 2 \cdot 0 \cdot y'(1) - 2 + y'(1) = 0,$$

that is, $y'(1) = 0$.

The full points of this part: 6 points:

(1) (4 points) Completely correct process of implicit differentiation deserves 4 points. If there are some minor algebraic mistakes in the process of implicit differentiation, one may gain at most 2 points.

(2) (2 points) Correct evaluation with the (right or wrong) obtained result of implicit differentiation to obtain $y'(1)$ deserves the rest 2 points. No partial credits will be given here.

(b) Applying the implicit differentiation twice, we have

$$\begin{aligned}&2\cos(xy) - 2x\sin(xy)(y + xy')\\ &- 2x\sin(xy)(y + xy') - x^2\cos(xy)(y + xy')^2 - x^2\sin(xy)(y' + y' + xy'')\\ &+ e^{y^2}(2yy')(2yy') + e^{y^2}2y' \cdot y' + e^{y^2}2yy''\\ &+ y''.\end{aligned}$$

Setting $x = 1$, $y(1) = 0$, and $y'(1) = 0$ we obtain that

$$2 - 0 - 0 - 0 - 0 + 0 + 0 + 0 + y''(1),$$

that is $y''(1) = -2$. The full points of this part: 6 points:

(1) (5 points) Completely correct process of second time implicit differentiation based on the result obtained in (1) deserves 5 points. If there are some minor algebraic mistakes in the process of implicit differentiation, one may gain at most 2 points.

(2) (1 points) Correct evaluation with the (right or wrong) obtained result of second time implicit differentiation to obtain $y''(1)$ deserves the rest 1 points. No partial credits will be given here.

(c) Since $y'(1) = 0$ and $y''(1) = -2 < 0$, $y(x)$ attains a local maximum at $x = 1$.

Based on the results obtained in (a) and (b) there are the following cases: if the obtained $y'(1) \neq 0$, the answer to (c) should be "No," which deserves 2 points. If the obtained $y'(1) = 0$ and $y''(1) < 0/ > 0$, the answer to (c) should be "$y(x)$ attains a local maximum/minimum at $x = 1$," which deserves 2 points. In the second case, the following situations will gain 1 point for partial credit:

(i) To conclude the monotonicity of y' without providing sufficient reasoning about the continuity of y''.

(ii) To conclude the concavity of y without providing sufficient reasoning about the continuity of y''.

3. (15 pts) Let

$$f(x) = \begin{cases} x^{\frac{1}{x}}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

(a) (6 pts) Use the definition of derivatives to compute $f'(0)$ as a limit.

(b) (9 pts) Write $f'(x)$ as a piecewise defined function. Is $f'(x)$ a continuous function over all real numbers?

Solution:

(a) By definition, $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$. It is clear that

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{0 - 0}{x - 0} = 0.$$

We claim that

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{x^{1/x}}{x} = \lim_{x\to 0^+} x^{\frac{1}{x}-1} = 0. \tag{1}$$

Let $g(x) = x^{\frac{1}{x}-1}$ and $h(x) = \ln g(x) = (\frac{1}{x} - 1)\ln x$. Since

$$\lim_{x\to 0^+} \frac{1}{x} - 1 = \infty \quad \text{and} \quad \lim_{x\to 0^+} \ln x = -\infty,$$

we have

$$\lim_{x\to 0^+} h(x) = -\infty,$$

and hence

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} e^{h(x)} = 0.$$

- Writing down the form of limit $\lim_{x\to 0^+} \dfrac{x^{1/x}}{x}$ or $\lim_{x\to 0^+} x^{\frac{1}{x}-1}$ deserves <u>2 points</u>.
- Showing $\lim_{x\to 0^+} x^{\frac{1}{x}-1} = 0$ correctly deserves <u>4 points</u>.

(b) It is clear that $f'(x) = 0$ if $x < 0$ and if $x = 0$ (by (a)), and this part deserves no points. We first compute $f'(x)$ for $x > 0$. Let $u(x) = \ln f(x) = \dfrac{\ln x}{x}$. Then, for $x > 0$, we have

$$\frac{f'(x)}{f(x)} = u'(x) = \frac{\frac{1}{x}\cdot x - (\ln x)\cdot 1}{x^2} = \frac{1 - \ln x}{x^2},$$

and hence

$$f'(x) = x^{\frac{1}{x}-2}(1 - \ln x) \quad \text{for every} \quad x > 0. \tag{2}$$

In summary, we have

$$f'(x) = \begin{cases} x^{\frac{1}{x}-2}(1 - \ln x) & \text{if} \quad x > 0; \\ 0 & \text{if} \quad x \leqslant 0. \end{cases}$$

Finally we show that $\lim_{x\to 0} f'(x) = 0 (= f'(0))$. Since $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0} 0 = 0$, it suffices to show that $\lim_{x\to 0^+} f'(x) = 0$, i.e.,

$$\lim_{x\to 0^+} x^{\frac{1}{x}-2}(1 - \ln x) = 0. \tag{3}$$

There are many different ways of showing this limit. For example, one may write $x^{\frac{1}{x}-2}(1-\ln x)$ as $x^{\frac{1}{x}-3}x(1-\ln x)$ and show that

$$\lim_{x\to 0^+} x^{\frac{1}{x}-3} = 0 \quad \text{and} \quad \lim_{x\to 0^+} x(1 - \ln x) = \lim_{x\to 0^+} x\ln x = 0:$$

for the first limit, one may proceed similarly as in the proof of (1) since

$$\lim_{x\to 0^+} \frac{1}{x} - 3 = \infty \quad \text{and} \quad \lim_{x\to 0^+} \ln x = -\infty,$$

and hence

$$\lim_{x\to 0^+} x^{\frac{1}{x}-3} = \lim_{x\to 0^+} \exp\big((\frac{1}{x} - 3)\ln x\big) = 0;$$

as for the second limit, we have

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} x = 0.$$

Note that here it is legitimate to apply the L'Hospital rule to compute the limit $\lim_{x\to 0^+} \frac{\ln x}{1/x}$ since $\lim_{x\to 0^+} 1/x = \infty$ and $1/x^2 \neq 0$ near $x = 0$. Here is another way of showing (3), adopted by one of the students. Consider the substitution $t = 1/x$. Then showing (3) is the same as showing $\lim_{t\to\infty} t^{-t+2}(1 + \ln t) = 0$, i.e.,

$$\lim_{t\to\infty} \frac{1 + \ln t}{t^{t-2}} = 0. \tag{4}$$

To see this, note that $\lim_{t\to\infty} t^{t-2} = \infty$ and that

$$\text{the derivative of } \ t^{t-2} \ \text{ is } \ t^{t-2}(\ln t + \frac{t-2}{t}),$$

which tends to $-\infty$ as $t \to \infty$, and hence $t^{t-2}(\ln t + \frac{t-2}{t}) \neq 0$ for t sufficiently large. Therefore one may apply the L'Hospital rule to obtain (4):

$$\lim_{t\to\infty} \frac{1 + \ln t}{t^{t-2}} = \lim_{t\to\infty} \frac{1/t}{t^{t-2}(\ln t + \frac{t-2}{t})} = 0.$$

- The full (2) deserves <u>4 points</u>.

(1) An intention of applying logarithmic differentiation to compute $f'(x)$ for $x > 0$ will gain 2 points.

(2) Finishing the process of logarithmic differentiation correctly will gain the rest 2 points.

- Showing (3) with correct reasoning deserves <u>5 points</u>. Partial credits may be given in the following situations:

(1) Trying to apply the L'Hospital rule to compute the limit in (3) with all conditions checked may gain at most 5 points.

(2) Trying to apply the L'Hospital rule to compute the limit in (3) without checking all conditions may gain at most 2 points.

(3) Some other unsuccessful attempts might gain at most 2 points, depending on the situations.

4. (14 pts) Consider the function $f(x) = 3x - \tan^{-1}(x-1)$.

(a) (6 pts) Show that the equation $3x - \tan^{-1}(x-1) = 3.01$ has a unique solution.

(b) (4 pts) Let $g(x)$ be the inverse function of f. Find $g(3)$ and $g'(3)$.

(c) (4 pts) Apply a linear approximation to g to get an estimate of the solution of $f(x) = 3.01$.

Solution:

(a) The function f is continuous and differentiable.

$f(0) = \frac{\pi}{4}, \quad f(1) = 3, \quad f(2) = 6 - \frac{\pi}{4}$

Intermediate value theorem: $f(1) < 3.01 < f(2)$ implies that there exist at least one solution.

If there are at least two solutions, then by Rolle's theorem, $f'(c) = 0$ for some c.

However, $f'(x) = 3 - \frac{1}{1+(x-1)^2} \geq 2$ is never equal to zero. Therefore there is exactly one solution.

(b) $g(3) = 1, g'(3) = \frac{1}{f'(1)} = \frac{1}{2}$

(c) $g(3.01) \approx 1 + \frac{1}{2}(3.01 - 3) = 1.005$

Grading scheme: $(6 + 4 + 4 = 14$ points)

(a) 3 points for existence and 3 points for only one solution. Basically all or nothing unless students make minor mistakes (example: quoting the wrong theorem but stating the condition and conclusion correctly. Just -1 in that case).

(b) 2 points each. All or nothing.

(c) 2 points for showing knowledge of linear approximation. 2 points for answer. If they got (b) wrong they can still get all 4 points here. But if they got (b) wrong but didn't follow through, then it depends on how they did the problem.

5. (24 pts) $f(x) = \frac{1}{x} e^{\frac{1}{x}}$ for $x \neq 0$.

(a) (7 pts) Compute $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$. Find vertical and horizontal asymptotes of $y = f(x)$.

(b) (7 pts) Compute $f'(x)$. Find critical point(s) of $f(x)$. Find interval(s) of increase and interval(s) of decrease of $y = f(x)$.

(c) (7 pts) Compute $f''(x)$. Discuss concavity of $y = f(x)$. Find inflection point(s), if any, of $y = f(x)$.

(d) (3 pts) Sketch the curve $y = f(x)$.

Solution:

(a) The right-hand limit is

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{1}{x} e^{\frac{1}{x}} = \infty, \qquad \textbf{(1 point)}$$

from which the curve $y = f(x)$ has a vertical asymptote $x = 0$ (**1 point**).

The left-hand limit is

$$\lim_{x\to 0-} f(x) \overset{y=\frac{1}{x}}{=} \lim_{y\to -\infty} \frac{y}{e^{-y}} \overset{\text{l'Hospital}}{=} \lim_{y\to -\infty} \frac{1}{-e^{-y}} = 0. \qquad \textbf{(2 points)}$$

Lastly, since

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{1}{x} e^{\frac{1}{x}} = 0 \qquad \textbf{(1 point)}$$

and

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{1}{x} e^{\frac{1}{x}} = 0, \qquad \textbf{(1 point)}$$

the curve $y = f(x)$ has one horizontal asymptote $y = 0$ (**1 point**).

(b) The first derivative of f is given by

$$f'(x) = -e^{\frac{1}{x}} \frac{x+1}{x^3}. \qquad \textbf{(3 points)}$$

So f has one critical point at -1 (**1 point**). By the increasing/decreasing test f is increasing on $(-1, 0)$ and decreasing on $(-\infty, -1)$ and $(0, \infty)$ (**3 points**).

(c) The second derivative of f is given by

$$f''(x) = e^{\frac{1}{x}} \frac{2x^2 + 4x + 1}{x^5}. \qquad \textbf{(3 points)}$$

By the concavity test f is concave upward on $\left(-1 - \frac{1}{\sqrt{2}}, -1 + \frac{1}{\sqrt{2}}\right) \cup (0, \infty)$ and concave downward on

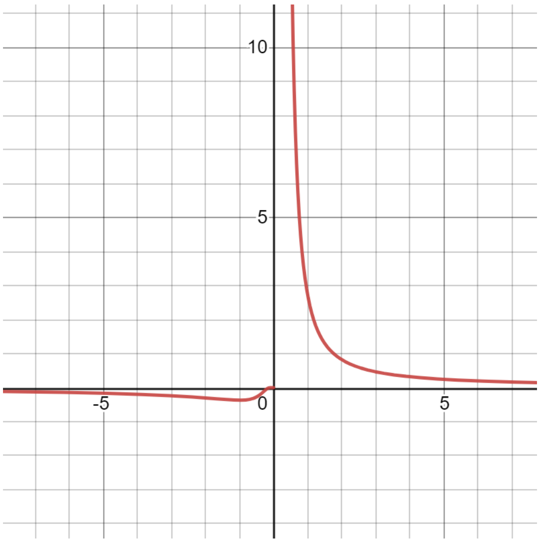
$$\left(-\infty, -1 - \frac{1}{\sqrt{2}}\right) \cup \left(-1 + \frac{1}{\sqrt{2}}, 0\right). \qquad \textbf{(3 points)}$$

Thus, f has inflection points at $-1 - \frac{1}{\sqrt{2}}$ and $-1 + \frac{1}{\sqrt{2}}$ (**1 point**).

(d) Vertical and horizontal asymptotes (**1 point**).

Intervals of increasing and decreasing (**1 point**).

Intervals of concave upward and downward (**1 point**).


10
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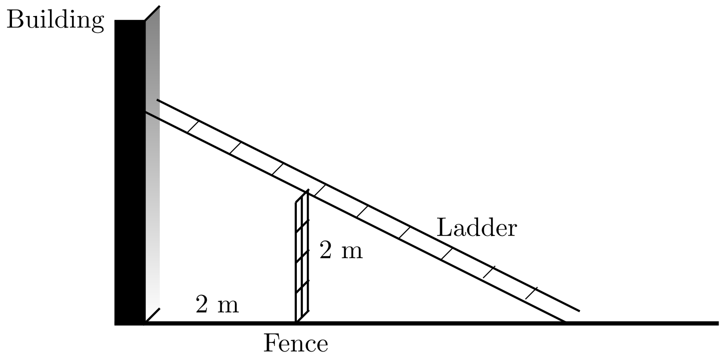
6. (12 pts) A fence 2 m tall is parallel to a tall building at a distance of 2 m from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?



Solution:

Method 1: θ is the angle between the ladder and the ground.

$$L(\theta) = 2\sec\theta + 2\csc\theta, \quad 0 < \theta < \pi/2$$

$$L'(\theta) = 2\sec\theta\tan\theta - 2\csc\theta\cot\theta$$

The only critical number in the domain satisfies

$$\tan^3\theta = 1, \quad \theta = \frac{\pi}{4}$$

$$L'(\pi/6) < 0, \quad L'(\pi/3) > 0$$

With this we verified that the critical number corresponds to the absolute minimum and the shortest length of the ladder would be $4\sqrt{2}$ m. □

Method 2: x is the distance from the base of the ladder to the fence.

$$L(x) = \sqrt{(x+2)^2 + \left(2 + \frac{4}{x}\right)^2}, \quad x > 0$$

$$L'(x) = \frac{x + 2 - \frac{4}{x^2}\left(2 + \frac{4}{x}\right)}{(x+2)\sqrt{1 + \frac{2}{x}}} = \frac{1 - \frac{8}{x^3}}{\sqrt{1 + \frac{2}{x}}}$$

The only critical number in the domain satisfies

$$x^3 = 8, \quad x = 2$$

$$L'(1) < 0, \quad L'(3) > 0$$

With this we verified that the critical number corresponds to the absolute minimum and the shortest length of the ladder would be $4\sqrt{2}$ m. □

Method 3: y is the height where the ladder touches the building.

$$L(y) = \sqrt{y^2 + \left(2 + \frac{4}{y-2}\right)^2}, \quad y > 2$$

Similar to above. □

Grading scheme: (12 points)

4 points for finding a function to optimize and state its domain.

4 points for solving for critical numbers.

3 points for verifying the critical number is a minimum.

1 point for stating the final answer.

Follow through applies to everything after the function. They only lose points later if their answer doesn't make sense.