1092 Calculus4 ME Final Exam

June 19, 2021

1. Let $V = \text{span}\left\{\begin{bmatrix}3\\-4\\-5\\-1\end{bmatrix}, \begin{bmatrix}4\\-2\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\3\\2\\3\end{bmatrix}, \begin{bmatrix}-2\\2\\-4\\1\end{bmatrix}\right\}$ and $W = \text{span}\left\{\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\-2\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\0\end{bmatrix}\right\}$.

   (a) (5 pts) Find the dimension of the vector subspace V.

   (b) (5 pts) Find a basis for W. (Hint: a basis is a linearly independent set of vectors that span W)

   (c) (2 pts) Show that V and W are not equal.

Solution:

(a) The dimension is the rank of the matrix $\begin{bmatrix}3&-4&-5&-1\\4&-2&-2&1\\1&3&2&3\\-2&2&-4&1\end{bmatrix}$. We find the rank of the matrix via Gaussian elimination.

$$\begin{bmatrix}3&-4&-5&-1\\4&-2&-2&1\\1&3&2&3\\-2&2&-4&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&3&2&3\\-2&2&-4&1\\3&-4&-5&-1\\4&-2&-2&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&3&2&3\\0&8&0&7\\0&-13&-11&-10\\0&-14&-10&-11\end{bmatrix}$$

$$\longrightarrow \begin{bmatrix}1&3&2&3\\0&1&0&\frac{7}{8}\\0&0&-11&\frac{11}{8}\\0&0&-10&\frac{10}{8}\end{bmatrix} \longrightarrow \begin{bmatrix}1&3&2&3\\0&1&0&\frac{7}{8}\\0&0&1&\frac{-1}{8}\\0&0&0&0\end{bmatrix}$$

The dimension of V is 3. □

Alternative methods for (a):

The determinant can be used to determine whether the matrix is full rank.

$$\det\begin{bmatrix}3&-4&-5&-1\\4&-2&-2&1\\1&3&2&3\\-2&2&-4&1\end{bmatrix} = 3\det\begin{bmatrix}-2&-2&1\\3&2&3\\2&-4&1\end{bmatrix} + 4\det\begin{bmatrix}4&-2&1\\1&2&3\\-2&-4&1\end{bmatrix} - 5\det\begin{bmatrix}4&-2&1\\1&3&3\\-2&2&1\end{bmatrix} + \det\begin{bmatrix}4&-2&-2\\1&3&2\\-2&2&-4\end{bmatrix}$$

$$= 3(-4-12-12-4-24+6)+4(8+12-4+4+48+2)-5(12+12+2+6-24+2)+(-48+8-4-12-16-8)$$

$$= -150 + 280 - 50 - 80 = 0$$

The rank of the matrix is 4 minus the dimension of the 0-eigenspace (which is found using Gaussian elimination). The eigenvectors for $\lambda = 0$ are $\begin{bmatrix}-5\\-7\\1\\8\end{bmatrix}t$, for all t. So the dimension of V is 3. □

They can also find the number of non-zero eigenvalues via the characteristic polynomial, which is $\lambda^4 - 4\lambda^3 + 34\lambda^2 - 11\lambda$. So the dimension of V is 3. $\square$

Here we also provide the Gaussian elimination without reordering if graders need to check for mistakes.

$$\begin{bmatrix} 3 & -4 & -5 & -1 \\ 4 & -2 & -2 & 1 \\ 1 & 3 & 2 & 3 \\ -2 & 2 & -4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{-4}{3} & \frac{-5}{3} & \frac{-1}{3} \\ 0 & 10 & 14 & 7 \\ 0 & 13 & 11 & 10 \\ 0 & -2 & -22 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{-4}{3} & \frac{-5}{3} & \frac{-1}{3} \\ 0 & 1 & 1.4 & 0.7 \\ 0 & 0 & -72 & 9 \\ 0 & 0 & -96 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{-4}{3} & \frac{-5}{3} & \frac{-1}{3} \\ 0 & 1 & 1.4 & 0.7 \\ 0 & 0 & 1 & \frac{-1}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\square$

And transposed.

$$\begin{bmatrix} 3 & 4 & 1 & -2 \\ -4 & -2 & 3 & 2 \\ -5 & -2 & 2 & -4 \\ -1 & 1 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & \frac{-2}{3} \\ 0 & 10 & 13 & -2 \\ 0 & 14 & 11 & -22 \\ 0 & 7 & 10 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & \frac{-2}{3} \\ 0 & 1 & 1.3 & -0.2 \\ 0 & 0 & -72 & -192 \\ 0 & 0 & 9 & 24 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & \frac{-2}{3} \\ 0 & 1 & 1.3 & -0.2 \\ 0 & 0 & 1 & \frac{8}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\square$

(b) We find a basis of W via Gaussian elimination.

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -2 & -2 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & -3 & -1 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & \frac{1}{3} & \frac{-2}{3} \\ 0 & 0 & 2 & 2 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & \frac{1}{3} & \frac{-2}{3} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis of W. Because we didn't switch the rows, $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ also form a basis of W. $\square$

Alternative methods for (b):

If we start by checking whether the four vectors are linearly independent by definition, we would be solving a system of equations.

$$\begin{cases} x + y + z - w = 0 \\ x - 2y + z + w = 0 \\ -x - 2y + z + 2w = 0 \\ -x + y + z = 0 \end{cases} \xrightarrow{w=x+y+z} \begin{cases} 2x - y + 2z = 0 \\ x + 3z = 0 \\ -x + y + z = 0 \end{cases} \xrightarrow{x=y+z} \begin{cases} y + 4z = 0 \\ y + 4z = 0 \end{cases}$$

So $x = 3,\ y = 4,\ z = -1,\ w = 6$ is a nonzero solution. The four vectors are linearly dependent. We can then check if three of them would be linearly independent.

$$\begin{cases} x + y + z = 0 \\ x - 2y + z = 0 \\ -x - 2y + z = 0 \\ -x + y + z = 0 \end{cases} \xrightarrow{x=y+z} \begin{cases} 2y + 2z = 0 \\ -y + 2z = 0 \\ -3y = 0 \end{cases}$$

Hence $x = 0, y = 0, z = 0$ is the only solution. The three vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ are linearly independent and form a basis of W. ☐

(c) From the results of the Gaussian elimination we can see that the vector $(0, 0, 1, 1)$ is in W but not in V. ☐

Alternative methods for (c):

Since both vector subspaces are 3-dimensional, we can check whether they have the same eigenvectors for $\lambda = 0$. The vector $\begin{bmatrix} -5 \\ -7 \\ 1 \\ 8 \end{bmatrix}$ is an eigenvector for V and the vector $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector for W, not parallel. ☐

We can also add a vector from V to W and show that the vector subspace would become 4-dimensional.

$$\left[\begin{array}{cccc} 1 & 1 & -1 & -1 \\ 1 & -2 & -2 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 1 & 3 & 2 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 1 & 1 & -1 & -1 \\ 0 & -3 & -1 & 2 \\ 0 & 0 & 2 & 2 \\ \hline 0 & 2 & 3 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 1 & 1 & -1 & -1 \\ 0 & -3 & -1 & 2 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 7 & 14 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 1 & 1 & -1 & -1 \\ 0 & -3 & -1 & 2 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 7 \end{array}\right]$$

☐

Grading:

(a) No points if the method is wrong. 2 points if they only showed that $\dim(V) \neq 4$ (by determinant). 1 point off for each minor mistake (algebra, miscopy). 2 points off for each major mistake (inventing row operations). Do not take more points off for wrong answer if it is a result of mistakes.

(b) 3 points for finding a basis. 2 points for whether their work showed linearly independence. Similar to (a), the work is more important than the answer.

(c) Depends on work shown in (a) and (b). Full credit or no credit.

Note:

For (a) and (b) if a minor mistake does not affect the answers, the grader may choose to take 0.5 off instead.

Because of the follow-through rule, they are allowed to say V and W are not equal due to dimension if they made a mistake in (a) or (b).

2. Let $A = \begin{bmatrix} -1 & 3 & 1 \\ 3 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) (8 pts) Find the eigenvalues of A and corresponding eigenvectors given that $\det(A - \lambda I_3) = -\lambda^3 - \lambda^2 + 12\lambda$.

(b) (4 pts) Diagonalize A. That is, find an orthogonal matrix P (i.e. $P^T P = I_3$) and a diagonal matrix D such that $P^T AP = D$.

(c) (4 pts) Determine whether $A + 5I_3$ is positive definite, negative definite, or indefinite.

(d) (4 pts) Determine whether $A - 5I_3$ is positive definite, negative definite, or indefinite.

Solution:

(a) $\det(A - \lambda I_3) = -\lambda^3 - \lambda^2 + 12\lambda = -\lambda(\lambda^2 + \lambda - 12) = -\lambda(\lambda - 3)(\lambda + 4)$. Eigenvalues are $0, 3, -4$. The eigenvectors corresponding to each:

$$\lambda = 0 \longrightarrow A - \lambda I_3 = \begin{bmatrix} -1 & 3 & 1 \\ 3 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{cases} -x + 3y + z = 0 \\ 3x - y + z = 0 \\ x + y + z = 0 \end{cases} \xrightarrow{x=3y+z} \begin{cases} 8y + 4z = 0 \\ 4y + 2z = 0 \end{cases} \longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} t$$

$$\lambda = 3 \longrightarrow A - \lambda I_3 = \begin{bmatrix} -4 & 3 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{cases} -4x + 3y + z = 0 \\ 3x - 4y + z = 0 \\ x + y - 2z = 0 \end{cases} \xrightarrow{x=-y+2z} \begin{cases} 7y - 7z = 0 \\ -7y + 7z = 0 \end{cases} \longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$$

$$\lambda = -4 \longrightarrow A - \lambda I_3 = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \longrightarrow \begin{cases} 3x + 3y + z = 0 \\ 3x + 3y + z = 0 \\ x + y + 5z = 0 \end{cases} \xrightarrow{z=-3x-3y} \left\{ -2x - 2y = 0 \right. \longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} t$$

□

(b) Hence if $P = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$, then P is orthogonal and $P^T AP = D$. □

(c) $A + 5I_3 = \begin{bmatrix} 4 & 3 & 1 \\ 3 & 4 & 1 \\ 1 & 1 & 6 \end{bmatrix}$ is Positive Definite. The eigenvalues of $A + 5I_3$ are $1, 5, 8$, all positive. □

They can also convert to $\mathbf{v}^T(A + 5I_3)\mathbf{v} = 4x^2 + 4y^2 + 6z^2 + 6xy + 2xz + 2yz$ and complete the squares:

$$4x^2 + 4y^2 + 6z^2 + 6xy + 2xz + 2yz = 3(x + y)^2 + (x + z)^2 + (y + z)^2 + 4z^2 \geq 0$$

□

Or use Sylvester's criterion for the matrix $A + 5I_3 = \begin{bmatrix} 4 & 3 & 1 \\ 3 & 4 & 1 \\ 1 & 1 & 6 \end{bmatrix}$.

$\det A_1 = 4 > 0$, $\det A_2 = 16 - 9 = 7 > 0$, $\det A_3 = 96 + 3 + 3 - 4 - 4 - 54 = 40 > 0$. □

(d) $A - 5I_3 = \begin{bmatrix} -6 & 3 & 1 \\ 3 & -6 & 1 \\ 1 & 1 & -4 \end{bmatrix}$ is Negative Definite. The eigenvalues of $A - 5I_3$ are $-9, -5, -2$, all negative. □

They can also convert to $\mathbf{v}^T(A - 5I_3)\mathbf{v} = -6x^2 - 6y^2 - 4z^2 + 6xy + 2xz + 2yz$ and complete the squares:

$$-6x^2 - 6y^2 - 4z^2 + 6xy + 2xz + 2yz = -3(x+y)^2 - (x+z)^2 - (y+z)^2 - 2x^2 - 2y^2 - 2z^2 \leq 0$$

□

Or use Sylvester's criterion for the matrix $A - 5I_3 = \begin{bmatrix} -6 & 3 & 1 \\ 3 & -6 & 1 \\ 1 & 1 & -4 \end{bmatrix}$.

$\det A_1 = -6 < 0$, $\det A_2 = 36 - 9 = 27 > 0$, $\det A_3 = -144 + 3 + 3 + 6 + 6 + 36 = -90 > 0$. □

Grading:

(a) 2 points for eigenvalues. 2 points for each corresponding eigenvector.

(b) 3 points for P and 1 point for D.

(c) 1 point for the correct conclusion. 3 points for justifying their result.

(d) 1 point for the correct conclusion. 3 points for justifying their result.

In general, 1 point off for each minor mistake (algebra, miscopy). 2 points off for each major mistake (inventing new math).

If they get a zero vector as eigenvector, then they get no points for the eigenvector AND lose points for (b).

3. Maximize $f(x, y, z) = yz$ subject to $x + z = 1$, $x^2 + y^2 \le 6$, $z \ge 0$.

(a) (4 pts) Check whether the NDCQ is satisfied.

(b) (8 pts) Write out the Lagrangian function and the first order conditions.

(c) (8 pts) Solve the constrained optimization problem given that the constraints form a closed and bounded region.

Solution:

(a) The equality constraint is $h(x, y, z) = x + z = 1$. The inequality constraints are $g_1(x, y, z) = x^2 + y^2 \le 6$ and $g_2(x, y, z) = -z \le 0$. $\nabla h = (1, 0, 1)$, $\nabla g_1(x, y, z) = (2x, 2y, 0)$, and $\nabla g_2(x, y, z) = (0, 0, -1)$ (2 pt)

If both g_1 and g_2 are binding, then $z = 0$, $x = 1$, and $y = \pm\sqrt{5}$. In this case, $\nabla h = (1, 0, 1)$, $\nabla g_1 = (2, \pm 2\sqrt{5}, 0)$ and $\nabla g_2 = (0, 0, -1)$ are linearly independent. (0.5 pt)

If only g_1 is binding, then $x^2 + y^2 = 6$ and $\nabla g_1(x, y, z) = (2x, 2y, 0) \ne (0, 0, 0)$. In this case, $\nabla h = (1, 0, 1)$ and $\nabla g_1 = (2x, 2y, 0)$ are linearly independent. (0.5 pt)

If only g_2 is binding, then $z = 0$, $x = 1$. In this case, $\nabla h = (1, 0, 1)$ and $\nabla g_2 = (0, 0, -1)$ are linearly independent. (0.5 pt)

If neither g_1 nor g_2 are binding, then we just need to check $\nabla h = (1, 0, 1)$ which is not $(0, 0, 0)$. (0.5 pt)

The above discussion shows that on the constraint set, ∇h and gradient(s) of binding inequality constraint(s) are linearly independent. Hence the NDCQ is satisfied.

(b) Lagrangian function $L(x, y, z, \mu, \lambda_1, \lambda_2) = yz - \mu(x + z - 1) - \lambda_1(x^2 + y^2 - 6) + \lambda_2 z$. (1 pt)

First order conditions:

$$L_x = -\mu - 2x\lambda_1 = 0 \;\; (1pt)$$
$$L_y = z - 2y\lambda_1 = 0 \;\; (1pt)$$
$$L_z = y - \mu + \lambda_2 = 0 \;\; (1pt)$$
$$\lambda_1 L_{\lambda_1} = \lambda_1(6 - x^2 - y^2) = 0 \;\; (1pt)$$
$$\lambda_2 z = 0 \;\; (1pt)$$
$$1 - x - z = 0, \; x^2 + y^2 \le 6, \; z \ge 0 \;\; (1pt)$$
$$\lambda_1 \ge 0, \; \lambda_2 \ge 0 \;\; (1pt)$$

□

(c) Since $f(x, y, z) = yz$ is zero when $z = 0$, we can assume $z > 0$ and $\lambda_2 = 0$.
(1 pt for ruling out the case $\lambda_2 > 0$.)

Now we have $\mu + 2x\lambda_1 = 0$, $2y\lambda_1 = z$, $y = \mu$, $x + z = 1$, $\lambda_1(6 - x^2 - y^2) = 0$.

If $\lambda_1 = 0$, then $x = 1, y = 0, z = 0, \mu = 0$ and $f(1, 0, 0) = 0$. (1 pt for the case $\lambda_1 = 0$.)

If $\lambda_1 \neq 0$, then $x^2 + y^2 = 6$. Replace μ and z to get $2x\lambda_1 + y = 0$, $2y\lambda_1 = 1 - x$. Hence $2xy\lambda_1 = -y^2 = x - x^2$. So we get $2x^2 - x - 6 = (2x+3)(x-2) = 0$.

If $x = 2$, then $z = -1$(not valid). If $x = \frac{-3}{2}$, then $z = \frac{5}{2}$. Then we get $y = \mu = \frac{\sqrt{15}}{2}$, $\lambda_1 = \frac{5}{2\sqrt{15}}$.
(If y is negative then $\lambda_1 < 0$.) The solution satisfies all our conditions.
(5 pts for the case $\lambda_1 > 0$ and the complete solution (x^*, y^*, z^*) and μ^*, λ_1.)

The maximum of $f(x,y,z) = yz$ is $\frac{5\sqrt{15}}{4}$ at $\left(\frac{-3}{2}, \frac{\sqrt{15}}{2}, \frac{5}{2}\right)$. (1 pt)

4. Suppose that you keep t hours a day as leisure time and $16-t$ hours to tutor with wage 400 dollars per hour. Your daily budget is $200+400(16-t)$ and you spend money on food and clothes with prices 250 and 350, respectively, per unit. If you consume x units of food and y units of clothes, then your utility function $U(x,y,t)$ depends on x, y and hours of leisure time t, where $\frac{\partial U}{\partial x} > 0$, $\frac{\partial U}{\partial y} > 0$, and $\frac{\partial U}{\partial t} > 0$. Now you want to maximize $U(x,y,t)$ under the constraints $250x+350y \le 200+400(16-t)$, $t \le 16$, $t \ge 0$, $x \ge 0$, $y \ge 0$.

(a) (8 pts) Write down the Kuhn-Tucker Lagrangian function, $\tilde{L}(x,y,t,\lambda_1,\lambda_2)$, and the first order conditions in the Kuhn-Tucker formulation.

(b) (4 pts) Show that if (x^*,y^*,t^*) is a maximizer, then the constraint $250x+350y \le 200+400(16-t)$ is binding at (x^*,y^*,t^*).

(c) (6 pts) Show that if (x^*,y^*,t^*) is a maximizer satisfying $x^* > 0$, $y^* > 0$, and $0 < t^* < 16$, then

$$\frac{\partial U}{\partial t}(x^*,y^*,t^*)\frac{1}{400} = \frac{\partial U}{\partial x}(x^*,y^*,t^*)\frac{1}{250} = \frac{\partial U}{\partial y}(x^*,y^*,t^*)\frac{1}{350}.$$

Solution:

(a) $\tilde{\mathcal{L}}(x,y,t,\lambda_1,\lambda_2) = U(x,y,t) - \lambda_1(250x + 350y - 200 - 400(16-t)) - \lambda_2(t-16)$ (1 pt)
At the maximizer (x^*,y^*,t^*), there are λ_1^* and λ_2^* s.t.

1. $\frac{\partial \tilde{L}}{\partial x} = \frac{\partial U}{\partial x}(x^*,y^*,t^*) - 250\lambda_1^* \le 0$
$\frac{\partial \tilde{L}}{\partial y} = \frac{\partial U}{\partial y}(x^*,y^*,t^*) - 350\lambda_1^* \le 0$
$\frac{\partial \tilde{L}}{\partial t} = \frac{\partial U}{\partial t}(x^*,y^*,t^*) - 400\lambda_1^* - \lambda_2^* \le 0$ (2 pts)
2. $x^*\frac{\partial \tilde{L}}{\partial x} = x^*(\frac{\partial U}{\partial x}(x_1^*,x_2^*,t^*) - 250\lambda_1^*) = 0$
$y^*\frac{\partial \tilde{L}}{\partial y} = y^*(\frac{\partial U}{\partial y}(x^*,y^*,t^*) - 350\lambda_1^*) = 0$
$t^*\frac{\partial \tilde{L}}{\partial t} = t^*(\frac{\partial U}{\partial t}(x^*,y^*,t^*) - 400\lambda_1^* - \lambda_2^*) = 0$ (2 pts)
3. $\frac{\partial \tilde{L}}{\partial \lambda_1} = -250x^* - 350y^* + 200 + 400(16-t^*) \ge 0$, $\frac{\partial \tilde{L}}{\partial \lambda_2} = -t^* + 16 \ge 0$ (1 pt)
4. $\lambda_1^*\frac{\partial \tilde{L}}{\partial \lambda_1} = \lambda_1^*(-250x^* - 350y^* + 200 + 400(16-t^*)) = 0$, $\lambda_2^*\frac{\partial \tilde{L}}{\partial \lambda_2} = \lambda_2^*(-t^*+16) = 0$ (1 pt)
5. $\lambda_1^* \ge 0$ and $\lambda_2^* \ge 0$ (1 pt)

(Students get full credits about the FOC with $(x,y,t),\lambda_1,\lambda_2$ instead of $(x^*,y^*,t^*),\lambda_1^*,\lambda_2^*$.)

(b) If $250x^* + 350y^* < 200 + 400(16-t^*)$, then $\lambda_1^* = 0$ (1 pt)
However, this implies that $\frac{\partial U}{\partial x}(x^*,y^*,t^*) \le 0$ and $\frac{\partial U}{\partial y}(x^*,y^*,t^*) \le 0$. (2 pts)
This contradicts the assumptions that $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial x} > 0$. (1 pt)
Hence we must have $250x^* + 350y^* = 200 + 400(16-t^*)$.

(c) Since $t^* < 16$, we have $\lambda_2^* = 0$ (1 pt).
Because $x^* > 0, y^* > 0$, we derive $\frac{\partial U}{\partial x}(x^*,y^*,t^*) - 250\lambda_1^* = 0$ and $\frac{\partial U}{\partial y}(x^*,y^*,t^*) - 350\lambda_1^* = 0$.

(2 pts)

Also, $0 < t^*$ implies that $\frac{\partial U}{\partial t}(x^*, y^*, t^*) - 400\lambda_1^* - \lambda_2^* = \frac{\partial U}{\partial t}(x^*, y^*, t^*) - 400\lambda_1^* = 0$ (1 pt)

Thus at the maximizer (x^*, y^*, t^*), $\frac{\partial U}{\partial x}\frac{1}{250} = \frac{\partial U}{\partial y}\frac{1}{350} = \frac{\partial U}{\partial t}\frac{1}{400} = \lambda_1^*$ (2 pts)

5. Consider the problem of maximizing $f(x, y, z) = xyz$ subject to $2x + y + z = 18$, and $x + 2y + z = 18$.

(a) (1 pts) Write down the Lagrangian function for this problem, $L(x, y, z, \mu_1, \mu_2)$, where μ_1 and μ_2 are the Lagrange multipliers.

Solution:

Grading scheme: 1 pt for the correct answer no partial credit for this part.
Solution: The Lagrangian function is

$$L(x, y, z, \mu_1, \mu_2) = xyz - \mu_1(2x + y + z - 18) - \mu_2(x + 2y + z - 18).$$

(b) (2 pts) Check whether the NDCQ is satisfied.

Solution:

Grading scheme: 0.75 pt for computing ∇h_1, 0.75 pt for computing ∇h_2, 0.5 pt for stating the rank is 2.
Solution: We have two constraint equalities $h_1(x, y, z) = 2x + y + z = 18$ and $h_2(x, y, z) = x + 2y + z = 18$. $\begin{bmatrix} \nabla h_1 \\ \nabla h_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. The rank of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ iis 2. So NDCQ is satisfied.

(c) (4 pts) Write down the first order conditions for this problem.

Solution:

Grading scheme: 0.8 pt for each part
Solution: We have to compute

$$\begin{aligned}
\frac{\partial L}{\partial x} &= yz - 2\mu_1 - \mu_2 = 0 \\
\frac{\partial L}{\partial y} &= xz - \mu_1 - 2\mu_2 = 0 \\
\frac{\partial L}{\partial z} &= xy - \mu_1 - \mu_2 = 0 \\
\frac{\partial L}{\partial \mu_1} &= -(2x + y + z - 18) = 0 \\
\frac{\partial L}{\partial \mu_2} &= -(x + 2y + z - 18) = 0
\end{aligned}$$

(d) (7 pts) Show that the solution of the first order conditions are $(x, y, z, \mu_1, \mu_2) = (4, 4, 6, 8, 8)$ or $(x, y, z, \mu_1, \mu_2) = (0, 0, 18, 0, 0)$. (You have to show your steps to get complete credits).

Solution:

Grading scheme: There are two solution. Each solution 3.5 point. Given partial credit accordingly.
Solution: We have $xyz - (2\mu_1 + \mu_2)x = xyz - (\mu_1 + 2\mu_2)y = xyz - (\mu_1 + \mu_2)z = 0$.
Using $2x + y + z = 18$ and $x + 2y + z = 18$, we add previous three equations to get

$$3xyz = \mu_1(2x + y + z) + \mu_2(x + 2y + z) = 18(\mu_1 + \mu_2).$$

From $xyz - (\mu_1 + \mu_2)z = 0$ and $xyz = 6(\mu_1 + \mu_2)$ we have $z = 6$ if $\mu_1 + \mu_2 \neq 0$. Now $2x + y = 12$ and $x + 2y = 12$. Then $x = 4$ and $y = 4$. Then $2\mu_1 + \mu_2 = 24$ and $\mu_1 + \mu_2 = 16$. Then $\mu_1 = 8$ and $\mu_2 = 8$. So $(x, y, z, \mu_1, \mu_2) = (4, 4, 6, 8, 8)$.

If $\mu_1+\mu_2=0$ then $xyz=0$. Hence $x=0$, $y=0$ or $z=0$. If $x=0$ then $\mu_1+2\mu_2=0, \mu_1+\mu_2=0$. This gives $\mu_1=\mu_2=0$. If $x=0$ then $y+z=18$ and $2y+z=18$. Thus $y=0$ $z=18$ Thus $(x,y,z,\mu_1,\mu_2)=(0,0,18,0,0)$.

If $y=0$ then $\mu_1=\mu_2=0$, $x=0, z=18$. Thus $(x,y,z,\mu_1,\mu_2)=(0,0,18,0,0)$ If $z=0$ then $\mu_1=\mu_2=0$, $y=6, x=6$. But this doesn't satisfy $xy=0$.

(e) (7 pts) Check the second order conditions at $(x,y,z,\mu_1,\mu_2)=(4,4,6,8,8)$ and $(x,y,z,\mu_1,\mu_2)=(0,0,18,0,0)$. Show that $(4,4,6)$ a local maximizer and $(0,0,18)$ is a local minimizer.

Solution:

Grading scheme: There are two solution. Each solution 3.5 point. Given partial credit accordingly.

Solution:

The constraint equations are $h_1(x,y.z)=2x+y+z=18$ and $h_2(x,y.z)=x+2y+z=18$ $\nabla h_1=(2,1,1)$ and $\nabla h_2=(1,2,1)$. $rank\begin{bmatrix}\nabla h_1\\ \nabla h_2\end{bmatrix}=rank\begin{bmatrix}2&1&1\\1&2&1\end{bmatrix}=2$. So NDCQ holds.

Recall

$$\frac{\partial L}{\partial x}=yz-2\mu_1-\mu_2=0$$
$$\frac{\partial L}{\partial y}=xz-\mu_1-2\mu_2=0$$
$$\frac{\partial L}{\partial z}=xy-\mu_1-\mu_2=0$$

We have

$$\frac{\partial^2 L}{\partial x^2}=0,\quad \frac{\partial^2 L}{\partial x\partial y}=z,\quad \frac{\partial^2 L}{\partial x\partial z}=y$$
$$\frac{\partial^2 L}{\partial y^2}=0,\quad \frac{\partial^2 L}{\partial y\partial z}=x,\quad \frac{\partial^2 L}{\partial z^2}=0.$$

So the bordered hessian matrix of Lagrangian is $H=\begin{bmatrix}0&0&2&1&1\\0&0&1&2&1\\2&1&0&z&y\\1&2&z&0&x\\1&1&y&x&0\end{bmatrix}$

At $(x,y,z,\mu_1,\mu_2)=(4,4,6,8,8)$, we have $H=\begin{bmatrix}0&0&2&1&1\\0&0&1&2&1\\2&1&0&6&4\\1&2&6&0&4\\1&1&4&4&0\end{bmatrix}$

Now $n=3$ $k=2$. $2k+1=5$ $n+k=5$.

We just need to compute $H_5=det(H)=det\begin{bmatrix}0&0&2&1&1\\0&0&1&2&1\\2&1&0&6&4\\1&2&6&0&4\\1&1&4&4&0\end{bmatrix}=det\begin{bmatrix}0&0&2&1&1\\0&0&1&2&1\\0&-1&-8&-2&4\\0&1&2&-4&4\\1&1&4&4&0\end{bmatrix}$

$=det\begin{bmatrix}0&2&1&1\\0&1&2&1\\-1&-8&-2&4\\1&2&-4&4\end{bmatrix}=det\begin{bmatrix}0&2&1&1\\0&1&2&1\\0&-6&-6&8\\1&2&-4&4\end{bmatrix}=(-1)\begin{bmatrix}2&1&1\\1&2&1\\-6&-6&8\end{bmatrix}=(-1)\begin{bmatrix}0&-3&-1\\1&2&1\\0&6&14\end{bmatrix}=$

$$\begin{bmatrix} -3 & -1 \\ 6 & 14 \end{bmatrix} = -36 < 0$$

$(-1)^k = 1$ $(-1)^n = -1$. So it is negative definite and it is a local maximizer.

At $(x, y, z, \mu_1, \mu_2) = (0, 0, 18, 0, 0)$, we have $H = \begin{bmatrix} 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 18 & 0 \\ 1 & 2 & 18 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$

$$detH = det \begin{bmatrix} 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 18 & 0 \\ 1 & 2 & 18 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} = det \begin{bmatrix} 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 18 & 0 \\ 0 & 1 & 18 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} = det \begin{bmatrix} 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 18 & 0 \\ 1 & 18 & 0 & 0 \end{bmatrix} = det \begin{bmatrix} 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 18 & 18 & 0 \\ 1 & 18 & 0 & 0 \end{bmatrix}$$

$$= (-1)det \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 18 & 18 & 0 \end{bmatrix} = (-18)det \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} = (-18)det \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= (-18)det \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = 36.$$

Now $k = 2$. $(-1)^k = 1 > 0$. So it is positive definite and it is a local minimizer.

(f) (1 pt) Does $f(x, y, z) = xyz$ have a global maximum or global minimum subject to $2x+y+z = 18$, and $x + 2y + z = 18$?

Solution:

Grading scheme: 0.3 point for finding the parametric equation of the line. 0.3 pt for finding f in one variable. 0.4 point to explain f goes to ∞ as the point goes to ∞.

Solution: The constraint set $2x + y + z = 18$, and $x + 2y + z = 18$ is a line. First, we have $2x + y + z - 2(x + 2y + z) = 18 - 36$, i.e. $-3y - z = -18$. Hence $z = -3y + 18$, $x = 18 - 2y - z = 18 - 2y + 3y - 18 = y$. Thus the constraint set can be expresses ad $x = y, z = -3y + 18$. $f(x, y, z) = y \cdot y \cdot (-3y + 18) = -3y^3 + 18y^2$ on the constraint set. $\lim_{y \to \infty} f = -\infty$ and $\lim_{y \to -\infty} f = \infty$. So f doesn't have a global maximum and a global minimum.

(g) (4 pts) Estimate the value of the local maximum of the following function $f(x, y, z) = xyz$ subject to $2x + y + z = 18.1$, and $x + 2y + z = 18.2$.

Solution:

Grading scheme: One point for setting up the right problem.

One point for pointing $\dfrac{\partial f(x^*(a_1, a_2), y^*(a_1, a_2), z^*(a_1, a_2))}{\partial a_1} = \mu_1^*(a_1, a_2)$

and one point for pointing $\dfrac{\partial f(x^*(a_1, a_2), y^*(a_1, a_2), z^*(a_1, a_2))}{\partial a_2} = \mu_2^*(a_1, a_2)$. One point for getting the right answer.

Solution: $(x^*(a_1, a_2), y^*(a_1, a_2), z^*(a_1, a_2), \mu_1^*(a_1, a_2), \mu_2^*(a_1, a_2))$ be the local maximizer and the multipliers of the following function.

Let $f(x, y, z, a) = xyz$ subject to $h_1(x, y, z) = 2x + y + z = 18 + a_1$ and $x + 2y + z = 18 + a_2$. We have

$$(x^*(0, 0), y^*(0, 0), z^*(0, 0), \mu_1^*(0, 0), \mu_2^*(0, 0)) = (4, 4, 6, 8, 8)$$

We have

$$\frac{\partial f(x^*(a_1,a_2),y^*(a_1,a_2),z^*(a_1,a_2))}{\partial a_1} = \mu_1^*(a_1,a_2) \text{ and}$$
$$\frac{\partial f(x^*(a_1,a_2),y^*(a_1,a_2),z^*(a_1,a_2))}{\partial a_2} = \mu_2^*(a_1,a_2).$$

Thus

$$\frac{\partial f(x^*(a_1,a_2),y^*(a_1,a_2),z^*(a_1,a_2))}{\partial a_1}\Big|_{(0,0)} = \mu_1^*(0,0) = 8 \text{ and}$$
$$\frac{\partial f(x^*(a_1,a_2),y^*(a_1,a_2),z^*(a_1,a_2))}{\partial a_2} = \mu_2^*(0,0) = 8.$$

Thus

$$f(x^*(0.1,0.2),y^*(0.1,0.2),z^*(0.1,0.2))$$
$$\approx f(x^*(0,0),y^*(0,0),z^*(0,0) + \mu_1^*(0,0)\cdot 0.1 + \mu_2^*(0,0)\cdot 0.2$$
$$= 96 + 0.8 + 1.6 = 98.4$$

(h) (4 pts) Estimate the value of the local maximum of the following function $f(x,y,z) = xyz+0.1x$ subject to $2x + y + 1.1z = 18$, and $x + 2y + z = 18.1$.

Solution:

Grading scheme: One point for setting up the right problem. One point for getting the Lagrangian $L(x,y,z,\mu_1,\mu_2,a) = xyz + ax - \mu_1(2x + y + (1 + a)z - 18) - \mu_2(x + 2y + z - 18 - a)$ One points for computing the right $\frac{\partial L}{\partial a}$. One point for getting the right answer.

Solution: Let $f(x,y,z,a) = xyz+ax$, $h_1(x,y,z,a) = 2x+y+(1+a)z = 18$ and $x+2y+z = 18+a$. Let $(x^*(a),y^*(a),z^*(a).\mu_1^*(a),\mu_2^*)$ be the local maximizer and the multipliers of the following function Let $f(x,y,z,a) = xyz + ax$ subject to $h_1(x,y,z,a) = 2x + y + (1 + a)z = 18$ and $x + 2y + z = 18 + a$.

We know that $(x^*(0),y^*(0),z^*(0),\mu_1^*(0),\mu_2^*(0)) = (4,4,6,8,8)$.

We also have $\frac{df((x^*(a),y^*(a),z^*(a),a)}{da} = \frac{\partial L}{\partial a}(x^*(a),y^*(a),z^*(a),\mu_1^*(a),\mu_2^*(a),a)$

Note that

$L(x,y,z,\mu_1,\mu_2,a) = xyz + ax - \mu_1(2x + y + (1 + a)z - 18) - \mu_2(x + 2y + z - 18 - a)$.

Then $\frac{\partial L}{\partial a} = x - \mu_1 z + \mu_2$.

So $\frac{df((x^*(a),y^*(a),z^*(a),a)}{da}|_{a=0} = x^*(0) - \mu_1^*(0)z^*(0) + \mu_2^*(0) = 4 - 8\cdot 6 + 8 = -36$.

Then

$$f(x^*(0.1),y^*(0.1),z^*(0.1),0.1)$$
$$\approx f(x^*(0),y^*(0),z^*(0),0) + \left(\frac{df((x^*(a),y^*(a),z^*(a),a)}{da}\Big|_{a=0}\right)\cdot 0.1$$
$$= 96 + (-36) * 0.1 = 92.8.$$